MATHEMATICAL NOTES- Constraints Theory **(Draft 4)**

0. Overview

The colour coding of the text in this document is defined as follows:

Black	Original Draft 3 text
Red	New text
Dark Red	New text about which we may wish to say more
Green	Possibly delete or abbreviate
Blue	Original Draft 3 text that needs more work

This mathematical discussion is in terms of scalar Hamiltonians for Classical Mechanics (CM) [14], [15] and Hermitian symmetric operator Hamiltonians for Quantum Mechanics (QM); these Hamiltonians are, ostensibly, independent of time. This avoids the complexities of the path integral method but restricts the argument to simple, isolated systems composed of particles that do not change or lose their identity. The discussion is thereby restricted to quantum systems that have a classical counterpart.

We begin with a classical archetype of point particles moving in a space P about which nothing is specified other than that it is continuous. The system of particles is associated with one or more functions θ of the coordinates $\underline{q}$. The function or functions θ is/ are assumed to be continuous and differentiable; and the coordinates are assumed to be continuous, differentiable functions of a single time measure t. It is possible to derive, from θ, an infinite set of scalar *differential identities*. From each of these an operator relation can be derived by a process of *quantization*. This process treats the scalar identities as if they were equations of motion. But the operator relations, called *constraints*, turn out *not* to be identities. As a consequence of the rules of quantization an operator analogue of Hamilton's equations holds.

These notes are concerned with the mathematical consequences of the constraints and their implications, if any, for physics. Typical of the questions asked are: Do the constraints apply to classical physics, quantum physics or both? What is the physical significance of the hierarchy of constraints? If, as appears likely, the constraints apply, principally, to classical physics can we recognise the laws of classical physics among their patterns? If so are there deviations, from the cannon laws, that might be interpreted as new physics? Are there internal inconsistencies in the constraints? Is it necessary to consider the whole of the infinite hierarchy of constraints, to describe classical physics, or is it sufficient to consider only a few of them? How reliable is our chosen method of quantization?

In summary: The constraints are shown to be related to the assumptions of continuity and differentiability that are necessary to derive the differential identities. The order of the derivatives in an identity fixes the position of the corresponding constraint in the hierarchy. Thus, the higher the level at which all the constraints are satisfied, the closer the operator description is to a *classical ideal*.

Given that θ is arbitrary the first constraint requires that the operator Hamiltonian is quadratic in the operators that are conjugate to the coordinate operators; this justifies reference to the former set as 'momenta'. The second constraint, when combined with the first, produces what looks like an operator field equation involving the operator Θ that represents θ; this is called the Operator Theta Equation. Functions of the coordinate operators that appear in the Hamiltonian form are candidates for Θ. In order that the spectra of coordinate velocities shall be continuous Θ cannot be a matrix; and it is this condition that ties the pattern of constraints to the laws of *classical physics*. It further appears that the first two constraints, only, are needed to describe the situations to which *classical mechanics* applies. We do not, however, have a definite theorem to this effect.

Reverse quantization of the general quadratic operator Hamiltonian and of the theta equation, produces recognisable expressions that belong to classical mechanics; these are called the Scalar Hamiltonian and the Scalar Theta Equation respectively. It is to be noted that, while the solutions of most conventional field equations satisfy appropriate versions of the scalar theta equation, not all solutions of that equation satisfy conventional equations. The reason is that the scalar theta equation, considered as a PDE, is of fourth order; new, or at any rate extra, physics is to be expected. The scalar Hamiltonian satisfies the classical Hamilton's equations.

Arbitrary functions of the coordinates $g^{uv} = g^{vu}$ appear as coefficients of the second order terms in the general scalar Hamiltonian. Hamilton's equations are recognisable as the generalised geodesic equations, with metric tensor $g^{uv} = g^{vu}$, of an higher dimension Riemannian space C. The coordinates of C are the aggregate $\underline{q}$ of the coordinates of all the particles contained in P. In consequence P is Riemannian; (C may be made to reduce to P under special circumstances). The geodesic equations, referred to C, describe the motion of a single representative point Q in C and, in consequence, the motions of all the particles in P.

Also, in the scalar Hamiltonian form, are arbitrary functions that, in ordinary physical space, would be described as gauge potentials. The rules of quantization require that P is flat and that the particle coordinates are orthogonal for each particle and flat; it follows that C is flat with flat coordinates.

Now consider a Riemannian space C' that is of the same dimension as C and may be curved with curvilinear coordinates $\underline{x}$. Suppose that C' is tangential to C at the point Q in C, with coordinates $\underline{q}$, and at the point X in C' with coordinates $\underline{x}$. Suppose also that the $\underline{x}$ are geodesic with pole X. Then, at the pole, we may choose $d\underline{q} = d\underline{x}$ and the functions g^{uv} may also be chosen as the metric tensor of C' with geodesic coordinates $\underline{x}$.

We now drop the prime on C' and assume that C may be curved; in the text we use the notation $\underline{q}$ for coordinates when C is flat and $\underline{x}$ for coordinates when C is curved.

The question then arises: Which tensor equation, with metric tensor $g^{uv} = g^{vu}$, reduces to the scalar theta equation when the coordinates are geodesic? The answer is the Kilmister Equation referred to here as the K equation. Notice that the K equation is not a quantum equation. It derives, ultimately, from a reverse quantisation of an operator Hamiltonian; it is, therefore classical. Various properties of C and the K equation can be deduced analytically. For example: the original time measure t is a geodesic distance in C; if the Ricci tensor of C vanishes then, the K equation is also satisfied, but there are solutions of the Kilmister equation for which the Ricci tensor is non-zero; when either the Ricci or Kilmister tensors vanish the dimension of C must be at least four for the space to be curved; because, in CT, it is axiomatic that each particle has only one time coordinate the remainder are space-like.

When the dimension of C is four, and there is only one particle in P, the K equation makes contact with *General Relativity* (GR). The resulting relativistic K equation makes no mention of matter; it is a set of up to ten PDEs that, given *four* sets of boundary conditions, describe the evolution of the metric tensor of C. Thus, through Einstein's equation, it describes a possible evolution of the matter-energy-momentum-stress tensor T_b^a. In the classical archetype, with which CT begins, the space between particles, in P, is assumed empty; but there seems to be no barrier to assuming that T_b^a describes a continuum. Notice that if space-time is truly empty then the relativistic K equation requires that Einstein's cosmological constant Λ is zero. Thus we are gambling; either the K equation describes the real world, at a classical level, or it does not!

The relativistic K equation is much more complicated than the Einstein equation. Detailed conclusions have only been deduced either by approximate analytic arguments, which assume low velocities and low curvature, or by machine calculations most of which are approximate in other ways.

The results of machine calculations can be summarised as follows: Analytic formulae are confirmed. With the *cosmological metric* (corresponding to an homogeneous, isotropic space-time) two *unique* and *exact* solutions are obtained; but they represent *empty* model universes. Various perturbations of these exact solutions represent model universes which are considered to be *well expanded*, contain uniform, *low density* distributions of *matter-energy* and have roughly constant Hubble parameters. All these model universes, once expanding, continue so to do.

We have modified the cosmological metric so that it is possible for it to describe weak gravity as a local perturbation. One class of these perturbation solutions expands at an almost constant rate, does not permit *Newtonian gravity* and may represent an early phase when the universe was filled only with radiation. The other class expands at a much lower rate, permits Newtonian gravity and has a matter-energy density ρ which is a sum of two terms ρ_0 and ρ_{00}. The term ρ_0 is the square of a real and so is positive. The term ρ_{00} can have either sign and arises because the K equation is of fourth order. The term ρ_0 is interpreted as ordinary mass density possibly including *Dark Matter*. The term ρ_{00} is interpreted as a *vacuum energy*; if $\rho_{00} < 0$ it could be an explanation of the *Dark Energy* effect. Note that the condensation of matter from an high density radiation field is a quantum phenomenon and thus outwith the competence of the K equation.

These machine solutions represent the local behaviour (less than 10^9 parsecs say) of well-expanded model universes; thus, if they have relevance to the actual universe, it cannot be to a very early phase. The cosmological metric is probably inappropriate to such a phase. So, in order to explore solutions of the K equation that might represent the early universe, we would need to choose a metrical form of an inhomogeneous, anisotropic space. This is an impediment to further progress.

We have studied flat or nearly flat spaces C by various means. Suppose that two particles in an E3 move, in a scalar potential ω, under Newtonian mechanics. Let the coordinates be Cartesian so that P is E3 with a unit diagonal metric. By writing out the classical Hamiltonian we see that C is a scaled E6 with a constant, diagonal metric. The potential ω is the only candidate for θ. The theta equation is therefore a PDE for the scalar potential ω. According to Einstein ω must be an invariant in P. The theta equation then reduces to

$$(0.1) \qquad \left(\frac{d^2}{dl^2} + \frac{2}{l}\frac{d}{dl} \right)^2 \omega = 0$$

where l is the distance between the particles. It follows that the only valid solution is

$$(0.2) \qquad \omega = \frac{c_1}{l} + c_2 l^2 + c_3 l + c_4; \quad c_1,....,c_4 \text{ are constants of integration}$$

The c_1 / l term gives an inverse square force along the line joining the particles; c_4 defines an arbitrary potential floor. The term $c_2 l^2 + c_3 l$ arises because the theta equation is of fourth order. It may be new physics; but to have evaded experimental detection it must be either very small, at laboratory distances, or the constants c_2 and c_3 must be zero. Notice that the above argument gives no clue as to whether ω is an electrostatic potential or a weak gravitational potential.

The generalisation of this argument considers $n_p > 2$ particles moving in a scalar field ω in E3. P is an E3 and C, given Cartesian coordinates, is a scaled $E3n_p$. Again ω is the only candidate for θ, is governed by the theta equation and must be an invariant in P. The theta equation, in this general case, is very much more complicated involving the distances $l_{\alpha\beta} = l_{\beta\alpha}$ between the α^{th} and β^{th} particles in P. For example when $n_p = 3$ the theta equation reduces to

$$\left(\frac{2}{l_{21}}\frac{\partial}{\partial l_{21}} + \frac{2}{l_{31}}\frac{\partial}{\partial l_{31}} + \frac{\partial^2}{\partial (l_{21})^2} + \frac{\partial^2}{\partial (l_{31})^2} + 2\cos\varphi_{3121}\frac{\partial^2}{\partial (l_{31})\partial (l_{21})}\right)^2 \omega = 0 \tag{0.3}$$

where

$$2l_{21}l_{31}\cos\varphi_{3121} = l_{21}^2 + l_{31}^2 - l_{32}^2 \tag{0.4}$$

It is obvious that

$$\omega = \frac{c_{21}}{l_{21}} + \frac{c_{31}}{l_{31}} + \frac{c_{32}}{l_{32}} + c_{00}; \quad c_{21}, c_{31}, c_{32}, c_{00} \text{ are constants} \tag{0.5}$$

is a particular solution of (0.3). This solution contains only terms that give the expected inverse square forces. But the general solution contains cross-terms. Provided that the only singularities are first order these cross-terms are contained in the regular part of the solution (0.5); and that is dominated by the singular part when the distances are small enough.

A more complicated calculation, in the $n_p = 2$ case, assumes that C is *slightly curved* and invokes the K equation to see what happens to the two particles. In order to allow comparison with previous calculations is assumed that the motions can be described, at least approximately, by a scalar potential ω. C has 8 dimensions and approximates a double Minkowski space. We connect ω to the metric tensor via the equations of motion in C (the geodesic equation and those involving derivatives of ω). We then express the Ricci tensor in terms of the derivatives of ω and approximate the K equation. The result is the same PDE as is obtained with the flat-space E6 Newtonian argument; it reduces to (0.1). This demonstration supports the notion that in order, at the least, to describe gravity C must be curved.

We have also studied the impact of CT on classical EM theory. The general quadratic scalar Hamiltonian, which is obtained from constraint 1, is compared with the classical Special Relativity (SR) Hamiltonian for a single charged particle moving in an EM field; $C \equiv P$ is chosen to be Minkowskian. The coefficients of the square terms in the CT Hamiltonian compare with those of the EM Hamiltonian; further, the coefficients of the linear terms in the CT Hamiltonian are related directly to the EM vector and scalar potentials.

CT has *no impact* on classical EM theory provided that the space between point, charged particles in P is considered *empty*. If a cloud of charged particles in P is represented, instead, by a charge density ρ and a current vector $\mathbf{i}$ then ρ and the components of $\mathbf{i}$ are subject to the wave equation.

As we climb the hierarchy of constraints their complexity increases hyper-exponentially. It is therefore very difficult to produce general theorems concerning their structure. This text contains some simple deductions. Irrespective of θ linear operator Hamiltonians, with constant scalar coefficients, satisfy *all* the constraints. The quadratic operator Hamiltonian P^2/m, where P is a Cartesian momentum and m is a scalar, also satisfies all the constraints given that θ satisfies the theta equation. The quadratic form

(0.6) $$H \equiv \sum_{j=1}^{n_c} \varepsilon_{jj} P_j^2; \quad \varepsilon_{jj} \equiv \pm 1$$

satisfies the constraints up to at least level 4 providing that θ satisfies the theta equation.

1. The Particle Space P - Differential Identities

We begin with a classical model of structureless point particles moving in a continuous space P. The coordinates $\underline{q} \equiv \{q^1, q^2, ...q^{n_c}, n_c \equiv n_p n_d\}$, where n_p is the number of particles and n_d the dimension of the space, are functions of a continuous time measure t; this measure is the time of the consciousness and the clock of a single observer [19]. We write an hierarchy of differential identities for $\dot{\theta}, \ddot{\theta}, ...;\quad \dot{\theta} \equiv d\theta/dt$ where $\theta(\underline{q})$ is any function of the coordinates characteristic of the model (e.g., a scalar potential, an element of a vector potential, an element of the fundamental tensor, a curvilinear coordinate). These are the scalar identities mentioned above. Note that the coordinates are numbered by superfixes in order to implement the Einstein summation convention [20].

Given a function $\theta(\underline{q})$ of the coordinates $\underline{q}(t)$ the first four of the infinite hierarchy of differential identities are:

(1.1) $$\dot{\theta} \equiv \frac{d\theta}{dt} = \dot{q}^j \theta_{,j}; \quad \theta_{,j} \equiv \frac{\partial \theta}{\partial q^j}$$

(1.2) $$\ddot{\theta} = \ddot{q}^j \theta_{,j} + \dot{q}^j \dot{q}^k \theta_{,jk}; \quad \theta_{,jk} \equiv \frac{\partial^2 \theta}{\partial q^j \partial q^k}$$

(1.3) $$\dddot{\theta} = \dddot{q}^j \theta_{,j} + 3\ddot{q}^j \dot{q}^k \theta_{,jk} + \dot{q}^j \dot{q}^k \dot{q}^l \theta_{,jkl};$$

(1.4) $$\ddddot{\theta} = \ddddot{q}^i \theta_{,i} + (4\dddot{q}^i \dot{q}^j + 3\ddot{q}^i \ddot{q}^j)\theta_{,ij} + 6\ddot{q}^i \dot{q}^j \dot{q}^k \theta_{,ijk} + \dot{q}^i \dot{q}^j \dot{q}^k \dot{q}^l \theta_{,ijkl};$$
$$i, j, k, l = 1, 2, ...n_c \equiv n_d n_p$$

The Einstein summation convention is in force; and, unless otherwise stated, all indices lie in the range $[1, n_c]$. Observe that the identities increase in complexity, very fast, as the order of the time derivative of θ rises. Kauffman has devised a recursive formula for the general case. This formula may be suitable for computer calculations.

2. Quantization

We quantize the identities by substituting Hermitian symmetrical operators, for scalar observables, according to the method of Schrödinger. The resulting expressions are then subject to non-commuting algebras [13], [21] which have application to both physics and mathematics.

The quantization rules require that P is Riemannian flat and that the coordinates $\underline{q}$ are also flat (the coordinates of a single particle are local Cartesian at every point in the space). It may be that these last two rules are not essential; but they turn out to be consistent with all that can be deduced from the other quantization rules.

2.1 Notations

Observables (denoted lower case) are represented by operators (denoted upper case). The only exception to this rule is the Hamiltonian which is always denoted upper case; the context indicates whether the CM scalar or QM operator is intended. If the observables are real then the operators have Hermitian symmetry and hence real spectra.

(2.1) $$a \to A; \quad \dot{a} \equiv \frac{da}{dt} \to \dot{A} \equiv \overset{1}{A}; \quad \frac{d^n a}{dt^n} \to \overset{n}{A}; \quad n = 1,2,....$$

where '→' means 'is represented by the operator'.

(2.2a) $$[A,B] \equiv AB - BA; \quad \lfloor A,B \rfloor \equiv \frac{i}{\hbar}[A,B]$$

(2.2b) $$\begin{aligned} &[A,B,C] \equiv [A,[B,C]]; \quad [A,B,C,D] \equiv [A,[B,C,D]]; \\ &\lfloor A,B,C \rfloor \equiv \lfloor A,\lfloor B,C \rfloor\rfloor; \quad \lfloor A,B,C,D \rfloor \equiv \lfloor A,\lfloor B,C,D \rfloor\rfloor \end{aligned}$$

(2.3) $$p_j \to P_j; \quad q^k \to Q^k; \quad A_{,j} \equiv \lfloor P_j, A \rfloor \quad A^{:k} \equiv \lfloor A, Q^k \rfloor$$

(2.4) $$\{A_1, A_2, ..., A_n\} \equiv \tfrac{1}{n!} \sum_{perm} A_1 A_2 .... A_n$$

where the commas on the LHS are inserted, if need be, only for clarity. The order of the arguments in $\{\}$ is immaterial. Notice that if an element inside any of the brackets $[.], \lfloor . \rfloor, \{\}$ is null then the bracket is null.

We combine these notations with the Einstein summation convention in a manner illustrated by the following examples:

(2.4a) $$\left[A^u, B_u^{vw}\right] \equiv \sum_{u=1}^{n_c} \left(A^u B_u^{vw} - B_u^{vw} A^u\right) \quad \left\lfloor A^u, B_u^{vw} \right\rfloor \equiv \frac{i}{\hbar}\left[A^u, B_u^{vw}\right]$$

(2.4b) $$\left\{A_x^{uvw}, A_{uv}^{yz}, A\right\} \equiv \frac{1}{3!} \sum_{u,v}^{n_c} \left(\sum_{perm} A_x^{uvw} A_{uv}^{yz} A \right)$$

where, in this example and for simplicity, A happens to be an invariant.

2.2 Schrödinger's Rules For Quantization

Schrödinger begins with a classical energy equation that expresses the total energy of a single particle system (the electron in the hydrogen atom) as the sum of its kinetic and potential energies

(2.5) $e \equiv \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + v(\underline{q})$; Newtonian approximation

He then replaces the scalars (energy e, Cartesian components of momentum p_1, p_2, p_3, Cartesian coordinates q_1, q_2, q_3 and the potential energy $v(\underline{q})$) by Hermitian symmetric operators

(2.6) $E \equiv H \equiv \frac{1}{2m}\left(P_1^2 + P_2^2 + P_3^2\right) + V(\underline{Q})$

In the Schrödinger coordinate representation (Q-diagonal) these operators are

(2.7)
$$E \equiv H \equiv i\hbar\frac{\partial}{\partial t}; \quad P_j \equiv -i\hbar\frac{\partial}{\partial q^j}; \quad Q^k \equiv q^k I \Rightarrow V(\underline{Q}) \equiv v(\underline{q})I$$
$$-\infty < p_j < \infty; \quad -\infty < q^k < \infty$$

where I is the unit operator and the mass m is taken as a c-number. He allows the first expression (2.7) to act on a wave function $\psi(\underline{q},t)$ regarded as a vector in an Hilbert space. The assumption that t is a scalar that commutes with all operators produces a PDE which can be solved for the function $\psi(\underline{q},t)$. The result is the Schrödinger Evolution.

(2.8) $i\hbar\frac{\partial\psi(\underline{q},t)}{\partial t} = H\psi(\underline{q},t) \Rightarrow \psi(\underline{q},t) = \exp(-iHt/\hbar)\psi(\underline{q},0)$

By substituting (2.6/7) on the RHS of the PDE he gets his famous wave equation

(2.9) $i\hbar\frac{\partial\psi(\underline{q},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + v(\underline{q})\right)\psi(\underline{q},t)$

2.3 Multiples, Powers, Sums And Products

From Schrödinger's rules we deduce the following generalizations: Suppose that

(2.10) $a \to A; \quad b \to B$

where a, b are real, scalar observables and A, B are their representative operators. Then for multiples, integer powers, sums (weighted by c-numbers α, β) and products

(2.11) $\alpha a^m \to \alpha A^m; \quad \alpha a + \beta b \to \alpha A + \beta B; \quad ab \to (AB + BA)/2; \quad m = 1,2,....$ [4]

whether or not A and B commute. From these formulae Kauffman deduces

(2.12) $\sum_{j=1}^{n} \alpha_j a_j \to \sum_{j=1}^{n} \alpha_j A_j; \quad a_1 a_2 .... a_n \to \frac{1}{n!} \sum_{perm} A_1 A_2 .... A_n \equiv \{A_1, A_2, ..., A_n\}$ [Kauffman]

whether or not the A_j mutually commute. Notice how, at (2.4a/b), this last formula may be combined with the summation convention.

2.4 Coordinates And Momenta

It follows from (2.7) that, in symbols,

(2.13) $P_j P_k = P_k P_j; \quad Q^j Q^k = Q^k Q^j; \quad Q^j P_k - P_k Q^j = i\hbar \delta^j_k I; \quad j,k = 1,2,3$

We assume, in addition, that these relations hold for all the coordinates of all the particles in P. That is

(2.14) $j,k = 1,2,....n_c$

This assumption takes us beyond Schrödinger's original rules; but it has been successful in models of multi-electron atoms.

Suppose that the operator Q^j has a continuous spectrum in the range $[-\infty, \infty]$; see (2.7). Then it can be proved that, if P_j satisfies (2.13) then, P_j is also an operator with a continuous spectrum in the range $[-\infty, \infty]$; so Q^j always has such a conjugate. What justification have we to call P_j a momentum operator representing a classical component of momentum? At this stage in the argument, none, save the success of Schrodinger's wave equation. Later we shall produce a more cogent argument.

2.5 The Schrödinger Representations And Time

In the Schrödinger representation the operators are independent of the time. From the rules so far given we can generalize as follows:

In the Q-diagonal Schrödinger representation

(2.15a) $q^j \to Q^j \equiv q^j I; \quad a(\underline{q}) \to A(\underline{Q}) \equiv a(\underline{q}) I; \quad -\infty < q^j < \infty; \quad I$ is the unit operator

(2.15b) $p_k \to P_k \equiv -i\hbar \frac{\partial}{\partial q^k}; \quad b(\underline{p}) \to B(\underline{P}) \equiv b(\underline{P})$

In the P-diagonal Schrödinger representation

(2.16a) $p^j \to P^j \equiv p^j I; \quad b(p) \to B(\underline{P}) \equiv b(\underline{p})I; \quad -\infty < p_j < \infty; \quad I$ is the unit operator

(2.16b) $q_k \to Q_k \equiv i\hbar \frac{\partial}{\partial p^k}; \quad a(\underline{q}) \to A(\underline{Q}) \equiv a(\underline{Q})$

Notice that in this representation the symbols $\underline{P}, \underline{Q}$ still satisfy (2.13) as they should. The connection between the Q-diagonal and P-diagonal Schrödinger representations is the multivariate Fourier transform.

Time is treated according to the rule

(2.17) $\dot{a} \to \dot{A} \equiv \lfloor H, A \rfloor$

2.6 The Heisenberg Representation And Time

In the Schrödinger representations the operators are independent of time and the state vectors depend on time. In the Heisenberg representations the operators depend on time and the state vectors are independent of time. The connection between a Schrödinger operator A and the corresponding Heisenberg operator $A(t)$ is

(2.18) $A(t) = U^+(t) A U(t); \quad U(t) \equiv \exp(-iHt/\hbar);$ '+' denotes Hermitian transpose

By differentiating (2.18) we get an Heisenberg representation of $\dot{a}$

(2.19) $\dot{A}(t) = \lfloor H, A(t) \rfloor = U^+(t) \lfloor H, A \rfloor U(t); \quad H(t) = H$; see (2.17)

This is consistent with (2.17) because

(2.20) $U^+(t)U(t) = U(t)U^+(t) = I; \quad I(t) = I$

or by setting $t = 0$ in (2.19) just as $A = A(0)$.

2.7 Derivatives With Respect To Coordinates And Momenta

Suppose that the operator A is a multinomial in the operators $\underline{P}, \underline{Q}$. What are the rules for calculating (see (2.2a))

(2.21) $A_{,j} \equiv \lfloor P_j, A \rfloor \equiv \frac{i}{\hbar}(P_j A - A P_j); \quad A^{:k} \equiv \lfloor A, Q^k \rfloor \equiv \frac{i}{\hbar}(A Q^k - Q^k A)$?

It can be shown that the rules can be summarized in the statements

$$(2.22)\quad A_{,j} = \frac{\partial A}{\partial Q^j}; \quad A^{:k} = \frac{\partial A}{\partial P_k}$$

as if $A, A_{,j}, A^{:k}, Q^j, P_k$ are scalars providing that the order of non-commuting operators is preserved; see (2.13). Now suppose that A is pure in the $\underline{P}$; that is, none of the $\underline{Q}$ appear in the expression for A. Then we deduce that $A^{:k}$ is pure in the $\underline{P}$ and is structured like the scalar partial derivative because all the terms in $A^{:k}$ commute

$$(2.23a)\ A^{:k}(\underline{P}) = \frac{\partial A(\underline{P})}{\partial P_k}$$

Similarly, if A is pure in the $\underline{Q}$ then $A_{,j}$ is pure in the $\underline{Q}$ and is structured like the scalar partial derivative because all the terms in $A_{,j}$ commute

$$(2.23b)\ A_{,j}(\underline{Q}) = \frac{\partial A(\underline{Q})}{\partial Q^j}$$

Finally, if A is pure (either in the $\underline{P}$ or the $\underline{Q}$) then, it can be proved, that the results (2.23) apply even when A is not a multinomial. It is necessary for the scalar partial derivatives $\partial a(\underline{p})/\partial p_k$ or $\partial a(\underline{q})/\partial q^j$, where $a \to A$, to exist. Kauffman calls commutators like (2.21) *derivations*.

Notice that

$$(2.23c)\ A_{,j,k} = A_{,k,j}; \quad A^{:j:k} = A^{:k:j}$$

are identities for any operator A. In particular, if A is pure in the $\underline{P}$ then,

$$(2.23d)\ A^{:j:k} = A^{:k:j} \Rightarrow a^{:jk} \equiv \frac{\partial^2 a}{\partial p_j \partial p_k} = \frac{\partial^2 a}{\partial p_k \partial p_j} \equiv a^{:kj}$$

using the P-diagonal Schrodinger representation and , if A is pure in the $\underline{Q}$ then,

$$(2.23e)\ A_{,j,k} = A_{,k,j} \Rightarrow a_{,jk} \equiv \frac{\partial^2 a}{\partial q_j \partial q_k} = \frac{\partial^2 a}{\partial q_k \partial q_j} \equiv a_{,kj}$$

using the Q-diagonal Schrodinger representation.

Now, by definition,

(2.24) $\theta(\underline{q}) \to \Theta(\underline{Q})$

where Θ is pure in the $\underline{Q}$. It follows that

(2.25) $\theta_{,ijk....} \to \Theta_{,i,j,k....}$; see (1.1….4) and (2.23b)

where, from (2.23c/…/e), the order of the suffices is immaterial. For example

(2.25a) $curl\theta = 0 \to curl\Theta = O$

are both identities.

2.8 Quantization Of The Differential Identities

We now have all the rules needed to quantize the differential identities (1.1….4). The quantizations of the first four identities, using the notations of Section 2.1, are:

(2.26) $Z_1 \equiv \left\{ H^{:j}, \Theta_{,j} \right\} = \lfloor H, \Theta \rfloor$

(2.27) $Z_2 \equiv \left\{ \lfloor H, H^{:j} \rfloor \Theta_{,j} \right\} + \left\{ H^{:j}, H^{:k}, \Theta_{,j,k} \right\} = \lfloor H, H, \Theta \rfloor$

(2.28) $$Z_3 \equiv \left\{ \lfloor H, H, H^{:j} \rfloor \Theta_{,j} \right\} + 3\left\{ \lfloor H, H^{:j} \rfloor H^{:k}, \Theta_{,j,k} \right\} + \left\{ H^{:j}, H^{:k}, H^{:l}, \Theta_{,j,k,l} \right\}$$
$$= \lfloor H, H, H, \Theta \rfloor$$

(2.29) $$Z_4 \equiv \left\{ \lfloor H, H, H, H^{:i} \rfloor \Theta_{,i} \right\} + 4\left\{ \lfloor H, H, H^{:i} \rfloor H^{j}, \Theta_{,i,j} \right\} + 3\left\{ \lfloor H, H^{:i} \rfloor \lfloor H, H^{:j} \rfloor \Theta_{,i,j} \right\}$$
$$+ 6\left\{ \lfloor H, H^{:i} \rfloor H^{:j}, H^{:k}, \Theta_{,i,j,k} \right\} + \left\{ H^{:i}, H^{:j}, H^{:k}, H^{:l}, \Theta_{i,j,k,l} \right\}$$
$$= \lfloor H, H, H, H, \Theta \rfloor$$

where, in particular, the operators (expressed in the Q-diagonal Schrödinger representation)

(2.30) $\Theta(\underline{Q}) \equiv \theta(\underline{q})I$; $\Theta_{,j} = \theta_{,j}I$; $\Theta_{,j,k} = \theta_{,jk}I$; $\Theta_{,j,k,l} = \theta_{,jkl}I$ etc.

are pure in the $\underline{Q}$. The Z_n are defined as the LHSs of the quantizations written in the above form; they are used below.

2.9 Apparent Ambiguity And Its Resolution- The First Quantization

The identity (1.1) is the sum of two-term products on the RHS

(2.31) $\dot{\theta} \equiv \dot{q}^j \theta_{,j}$

So the RHS of its quantization

(2.32) $\lfloor H, \Theta \rfloor = \{H^{:j}, \Theta_{,j}\}$; see (2.26)

is, in our notation, the two-term curly bracket with its implied summation. This simple structure arises because we do not enquire as to the form of the Hamiltonian either as a classical scalar with

(2.33) $\dot{q}^j = \dfrac{\partial H}{\partial p_j}$

or as an Hermitian operator with

(2.34) $\dot{Q}^j = H^{:j}$

Suppose, for example, that we choose the general quadratic form

(2.35a) $H \equiv \{G^{uv}(\underline{Q}), P_u, P_v\} + \{F^j(\underline{Q}), P_j\} + V(\underline{Q})$; $G^{uv} = G^{vu}$; G^{uv}, F^j, V Hermitian

with a reverse quantization

(2.35b) $H \equiv g^{uv}(\underline{q}) p_u p_v + f^j(\underline{q}) p_j + v(\underline{q})$

Scalar (2.35b) is the classical analogue of operator (2.35a). If we quantize (2.35b) we should get (2.35a). But there may be ambiguity. For we might quantize (2.35b) as

(2.35c) $H \equiv \{G^{uv}(\underline{Q}), P_u P_v\} + \{F^j(\underline{Q}), P_j\} + V(\underline{Q})$; the P_u mutually commute

regarding $p_u p_v$ as a *single scalar*. But the operator $\{G^{uv}, P_u, P_v\}$ may differ from the operator $\{G^{uv}, P_u P_v\}$. We can analyse this situation as follows: According to the rules (2.4b/12) product

(2.36) $abc \to \{A, B, C\} = \dfrac{1}{3!}\left[ABC + ACB + BAC + BCA + CAB + CBA\right]$

where a, b, c are real scalars and A, B, C are Hermitian operators. But if

(2.37) $AB = BA,\ AC \neq CA,\ BC \neq CB,\ (AB)C \neq C(AB)$

then we might say

(2.38) $(ab)c \rightarrow \{AB, C\}$

For this to be consistent with (2.36/37)

$$(2.39)\quad \frac{1}{6}\left[2ABC + ACB + BCA + 2CAB\right] = \frac{1}{2}\left[ABC + CAB\right]$$
$$\Rightarrow A(CB - BC) = CAB - BCA$$

As a general result (2.39) looks most unpromising; but, if we apply it to (2.35a/c), we find that

(2.40) $\{G^{uv}, P_u, P_v\} = \{G^{uv}, P_u P_v\}$; summed

only if

(2.41) $-\frac{\hbar}{i} P_u G^{uv}_{,v} = G^{uv} P_u P_v - P_v G^{uv} P_u$; summed; see (2.39)

But this is an identity; see Section 2.7. So (2.40) is always true and it does not matter which of the quantizations (2.35a/c) we arrive at from (2.35b).

Returning to the quantization (2.31/32): we note that with the classical Hamiltonian (2.35b) we have

(2.42a) $\dot{q}^j = \frac{\partial H}{\partial p_j} = 2g^{uj} p_u + f^j$

with quantization

(2.42b) $\dot{Q}^j = 2\{G^{uj}, P_u\} + F^j$

Alternatively, with the operator Hamiltonian (2.35c), we may use the rules of Section 2.7 to arrive directly at

(2.42c) $\dot{Q}^j = H^{:j} = 2\{G^{uj}, P_u\} + F^j$; see (2.34).

Given (2.42a) the identity (2.31) becomes

(2.43a) $\dot{\theta} = \dot{q}^j \theta_{,j} = \left(2g^{uj} p_u + f^j\right)\theta_{,j}$

with quantization

(2.43b) $\lfloor H,\Theta \rfloor = 2\{\Theta_{,j}, G^{uj}, P_u\} + \{F^j, \Theta_{,j}\}$

Applying (2.39)

(2.44a) $\{\Theta_{,j}, G^{uj}, P_u\} = \{\Theta_{,j} G^{uj}, P_u\}$

only if

(2.44b) $\frac{\hbar}{i} G^{uj}_{,u} = P_u \Theta_{,j} G^{uj} - G^{uj} P_u \Theta_{,j}$

which turns out to be an identity. But it also turns out that, given (2.42c), the last equation

(2.45) $\{H^{\cdot j}, \Theta_{,j}\} = \{2\{G^{uj}, P_u\} + F^j, \Theta_{,j}\} = 2\{\Theta_{,j} G^{uj}, P_u\} + \{F^j, \Theta_{,j}\}$

is an identity. Thus the quantization (2.31/32) is consistent with (2.35a/c), (2.42b/c) and (2.43a/b) however arrived at; (i.e., either by quantizing classical expressions or by applying operator algebra according to the above rules).

We have shown that, given the quadratic Hamiltonians (2.35a/c), apparent ambiguities, associated with the simplest quantization (2.26), are illusory. We must always keep in mind, however, the possibility that the higher level quantizations (2.27) et seq. might be either ambiguous or inconsistent.

2.10 Poisson Brackets- An Alternative Method Of Quantization

Suppose that $f(\underline{p},\underline{q},t)$ is a classical dynamical variable associated with the system of particles in P. Then

(2.46) $$\begin{aligned}\dot{f} &= \frac{\partial f}{\partial p_j}\dot{p}_j + \frac{\partial f}{\partial q^k}\dot{q}^k + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_j}\frac{\partial H}{\partial q^j} + \frac{\partial f}{\partial t}\end{aligned}$$; summation convention in force; $j,k = 1,2,\dots,n_c$

where H is the classical Hamiltonian. The quantity

(2.47) $$PB(f,H) \equiv \frac{\partial f}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_j}\frac{\partial H}{\partial q^j} = -PB(H,f)$$

is called the *Poisson Bracket* of f and H [5, p. 86]. Quantizing (2.47) by the rules of Sections 2.2/…/7, on the assumption that f does not depend explicitly on t, (2.48a) $\lceil F, H \rceil \equiv \{F_{,k}, H^{:k}\} - \{F^{:j}, H_{,j}\}$

where H is now the QM Hamiltonian operator and

(2.48b) $f \to F; \quad PB(f, H) \to \lceil F, H \rceil$; definition

On the further assumption that $f(\underline{q}) \to F(\underline{Q})$ is pure in the $\underline{q} \to \underline{Q}$

(2.49a) $F^{:j} = O \Rightarrow \lceil F, H \rceil = \{F_{,k}, H^{:k}\}$; see (2.48a)

Thus, if (2.26) is satisfied for $\theta \equiv f$ then,

(2.49b) $\lceil F, H \rceil = \lfloor H, F \rfloor$

That is the operator $\lceil F, H \rceil$, corresponding to the classical Poisson bracket $PB(f, H)$, is equal to the QM commutator $\lfloor H, F \rfloor$.

More generally, if $x(\underline{p}, \underline{q}, t)$ and $y(\underline{p}, \underline{q}, t)$ are any two classical dynamical variables then, the quantity

(2.50) $$PB(x, y) \equiv \frac{\partial x}{\partial q^k}\frac{\partial y}{\partial p_k} - \frac{\partial x}{\partial p_j}\frac{\partial y}{\partial q^j}$$

is called the *Poisson Bracket* (PB) of x and y [5, p. 86]. Dirac [1, p. 85] asserts that if

(2.51) $x \to X; \quad y \to Y; \quad PB(x, y) \to \lceil X, Y \rceil$; definition

then

(2.52) $\lceil X, Y \rceil = \lfloor Y, X \rfloor$; see (2.49)

as a general proposition. Now if

(2.53a) $x \equiv p_j; \quad y \equiv q^k$

then

(2.53b) $PB(x, y) = -\delta_{jk}$

So, according to (2.52),

(2.54) $\lceil P_j, Q^k \rceil = -\lfloor P_j, Q^k \rfloor = -\delta_{jk} I$; see (2.13)

which is the correct quantization of (2.53b) according to (2.13). Similarly we obtain the rest of the rules (2.13) by applying (2.52).

With $n_c = 3$ we can define the familiar components of classical angular momentum

(2.55a) $l_1 \equiv q^2 p_3 - q^3 p_2; \quad l_2 \equiv q^3 p_1 - q^1 p_3; \quad l_3 \equiv q^1 p_2 - q^2 p_1$; see [5, p. 89]

with quantizations, according to the rules of Sections 2.2/…/7,

(2.55b) $L_1 \equiv Q^2 P_3 - Q^3 P_2; \quad L_2 \equiv Q^3 P_1 - Q^1 P_3; \quad L_3 \equiv Q^1 P_2 - Q^2 P_1$

We find

(2.56) $PB(l_i, l_j) = \sum_k \varepsilon_{ijk} l_k; \quad \varepsilon_{ijk}$ is the usual permutation symbol ; see (2.50)

So, according to (2.52), the quantization of (2.56) should be

(2.57) $\lceil L_i, L_j \rceil = \sum_k \varepsilon_{ijk} L_k = -\lfloor L_i, L_j \rfloor$

The last of the equations (2.56) is satisfied by (2.55b) using the rules of Sections 2.2/…/7. All the usual results, for the quantization of angular momentum, follow from either method. For example

(2.58) $\lfloor L^2, L_i \rfloor = \sum_{j,k} 2\varepsilon_{ijk} L_j L_k = O; \quad L^2 \equiv \sum_i L_i^2$

Clearly (2.50/52) tell us about the quantization of derivatives. Setting

(2.59) $x \equiv x(\underline{q}); \quad y \equiv p_j \Rightarrow PB(x, y) = \dfrac{\partial x(\underline{q})}{\partial q^j}$

So, according to (2.52), the quantisation of the derivative of x pure in the $\underline{q}$ is

(2.60) $\dfrac{\partial x(\underline{q})}{\partial q^j} \rightarrow \lfloor P_j, X \rfloor \equiv X_{,j}$

Similarly, setting

$$(2.61)\quad x \equiv q^k; \quad y \equiv y(\underline{p}) \Rightarrow PB(x,y) = \frac{\partial y(\underline{p})}{\partial p_k},$$

So, according to (2.52), the quantisation of the derivative of y pure in the $\underline{p}$ is

$$(2.62)\quad \frac{\partial y(\underline{p})}{\partial p_k} \to \lfloor Y, Q^k \rfloor \equiv Y^{:k}$$

The results (2.60/62) are exactly the rules (2.3/23).

Despite the above agreements between the two methods they are disparate. We see this clearly in the first example given at (2.49b). The result is true only if quantization 1 is satisfied with $\theta \equiv f$. Now there are other, independent and compelling reasons why quantization 1 should be satisfied for arbitrary θ; see Section 3.1 below. Nevertheless, without this condition, the method which replaces PBs by commutators gives results which differ from (2.49a). The reason is that this method does not give explicit rules for sums, powers and products; see Section 2.3. In the deduction of rule (2.52) both [1] and [5] follow the same argument: The properties of PBs are taken to be summarised by

$$PB(x,y) = -PB(y,x); \quad PB(x,x) = 0;$$
$$(2.63)\quad PB(x, y+z) = PB(x,y) + PB(x,z); \quad PB(x, yz) = yPB(x,z) + zPB(x,y);$$
$$PB(q_i, p_j) = \delta_{ij}; \quad PB(p_i . p_j) = 0 = PB(q_i . q_j)$$

It is then assumed that the Hermitian operators that appear in the definitions

$$(2.64)\quad x \to X; \quad y \to Y; \quad z \to Z; \quad PB(x,y) \to \lceil X, Y \rceil; \text{ definition}$$

satisfy rules that *look similar* to the formulae (2.63). In particular

$$(2.65)\quad \lceil X, YZ \rceil = Y\lceil X, Z \rceil + \lceil X, Y \rceil Z; \quad \lceil XY, Z \rceil = X\lceil Y, Z \rceil + \lceil X, Z \rceil Y$$

where care has to be taken to preserve order because of non-commutativity. It is then deduced that, for any four Hermitian operators W, X, Y, Z,

$$(2.66)\quad [Y, W]\lceil X, Z \rceil = \lceil W, Y \rceil [X, Z]$$

Because the four operators are arbitrary (2.66) is an identity only if, for any two operators, A, B

$$(2.67)\quad [A, B] = \alpha \lceil A, B \rceil$$

where α is a scalar constant (which commutes with all operators). In order that (2.54) should be satisfied with

(2.68) $A \equiv P_j; \quad B \equiv Q^k$

we must put

(2.69) $\alpha = i\hbar$

It is remarkable that (2.65) leads to (2.67) and thence to (2.52); it is also remarkable that (2.52) gives so many agreements with the Schrodinger method of Sections 2.2/…/7. But the step from (2.63) to (2.65) is 'sleight of hand' almost without rational basis. In both [1] and [5] this sleight of hand is covered up by a disreputable use of notation. We conclude that the PB method of quantization is not a viable alternative to the Schrodinger method.

2.11 Feynman's Path Integral Method

The Feynman path integral method of quantisation must receive brief mention. The method has been an important instrument in the creation of the Standard Model [23]. The Feynman method of quantization expresses the state, at a given time, by means of a path integral (sum of histories) from a previous time [24]. The kernal of this integral involves the exponent of a scalar Lagrangian that describes the system. The difficulty of applying the Feynman method to CT is the difficulty of generating operator constraints, or something like them, from the path integral; if this could be done the structure of the Lagrangian would, presumably, be linked to the constraints. No such argument has been found.

3. Constraints And Constraint Theory- Preliminaries

The quantizations, unlike the differential identities for $\dot{\theta}, \ddot{\theta}, \ldots$ etc., are not, in general, identities. They restrict the forms permitted for θ and the operator H; they are *constraints*. The constraints force patterns/ regularities on P and on the behaviour of the particles within it. A Constraint Theory (CT) is thus defined. The object of the theory is to study the patterns produced by the constraints and to see if those patterns describe recognizable physics. If they do so describe there is then a further topic of investigation defined by the question: Are there implied extensions, to conventional physics, which lead to testable results?

We are gambling! If the rules of quantization are in error then the results of the theory may be nonsense; and, in any case, there may be alternative rules. But, if the rules of quantization are correct then, the regularities expressed by the constraints are thought to explain CM in terms of QM [21].

3.1 The Significance Of The Hierarchy Of Constraints

The infinite hierarchy of *differential identities* exemplified by (1.1/.../4) is an expression of a classical ideal: namely, that all the variables are continuous and all the functions of those variables are continuous and of class C^∞; in consequence the space P is continuous. A system that conforms to this ideal is smoothly deterministic. If any of the differential identities fail then one or more of the variables, the functions or their derivatives must be discontinuous; such failure could also be caused by a discontinuity in P.

The *quantizations* are each the direct consequence of the above substitution rules applied to a differential identity. Conversely if we reverse a quantization, by reversing each of the applicable rules, then we obtain the corresponding differential identity. It follows that if one or more of the quantizations does not hold, through injudicious choice of H and $\theta \to \Theta$, then the corresponding differential identities do not hold and the classical ideal is compromised. In short: the infinite hierarchy of constraints is a QM expression that the underlying classical model is ideal. The following argument shows that the satisfaction of the constraints has to do also with predictability.

Because the $\underline{q}$ are functions of the time t the scalar function $\theta(\underline{q})$ can, in principle, be expanded as a function of t. The resulting series can be quantized in terms of operators in the Schrödinger representation

$$(3.1)\quad \theta(t) = \theta_0 + \dot{\theta}_0 t + \ddot{\theta}_0 t^2/2 + ... \to \Theta(t) = \Theta + \dot{\Theta} t + \ddot{\Theta} t^2/2 + ...; \quad \theta_0 \equiv \theta(\underline{0})$$

As we have seen the Schrödinger operators $\dot{\Theta}, \ddot{\Theta}, ...$ are generated according to the rule

$$(3.2)\quad \left.\frac{d^s\theta}{dt^s}\right|_{t=0} \to \overset{s}{\Theta} \equiv \left\lfloor H, \overset{s-1}{\Theta} \right\rfloor; \quad \Theta \equiv \overset{0}{\Theta}; \quad s = 1,2,...; \text{ see (2.17)}$$

and so (3.1) becomes

$$(3.3)\quad \theta(t) \to \Theta(t) = \exp(iHt/\hbar)\Theta\exp(-iHt/\hbar)$$

which is the Heisenberg representation of θ; see (2.18). Writing the constraints in the form

$$(3.4)\quad Z_s = \overset{s}{\Theta}; \quad s = 1,2,...; \text{ see (2.26…29)}$$

the formula (3.1) can be expressed as

$$(3.5)\quad \Theta(t) = \Theta + \sum_{j=1}^{\infty} \frac{Z_j t^j}{j!}$$

Thus the constraints are expressed by equating the coefficients of powers of t across (3.1/5).

Now suppose that all the constraints are satisfied up to a finite level n; that is equation (3.4) is satisfied only for $s \le n$. Then $\Theta(t)$ is approximated by

$$(3.6a)\quad \Theta(t) \approx \Theta + \sum_{j=1}^{n} \frac{Z_j t^j}{j!}$$

Unless the series (3.1) terminates this approximation will be valid only for sufficiently small t. Clearly, in general, the higher the level n, to which all constraints are satisfied, the better the approximation for a given t. Put another way: the higher n the better the RHS of (3.6) is as a predictor of $\Theta(t)$ and its expectation

$$(3.6b)\quad \langle \Theta(t) \rangle = \langle t | \Theta | t \rangle \approx \langle \Theta \rangle + \sum_{j=1}^{n} \frac{\langle Z_j \rangle t^j}{j!}$$

Similarly, if the quantities $\overset{s}{\theta}_0$, $s = 1,2,....$ are known then the first equation (3.1) can be used to make a classical prediction of $\theta(t)$. When n is finite, however, levels $n+1, n+2,....$ of the differential identities may not hold. It follows that only the formula

$$(3.6c)\quad \theta(t) \approx \theta_0 + \sum_{j=1}^{n} \frac{\overset{j}{\theta}_0 \, t^j}{j!},$$

with no extra terms, can be relied upon.

Thus, if n is finite, additional uncertainties are added to the usual QM uncertainties. Because $\hbar$ is small this situation may be summarised by the dictum: "satisfaction of higher level constraints forces the behaviour towards a classical ideal" (Kauffman).

No prediction, of any kind, is possible unless $n = 1$; and the above argument suggests that $n > 1$ is always desirable.

3.2 Constraint 1 Requires A Polynomial Form For H With Order At Most 2

The first of the constraints is

$$(3.7)\quad \{H^{;j}, \Theta_{,j}\} = \lfloor H, \Theta \rfloor; \text{ see (2.26)}$$

Written in full this is

(3.8) $\frac{i}{2\hbar}\left((HQ^k - Q^k H)\Theta_{,j} + \Theta_{,j}(HQ^k - Q^k H)\right) = \frac{i}{\hbar}(H\Theta - \Theta H)$

We now consider what happens to H when $\theta(\underline{q})$ is arbitrary. Suppose that $n_c = 1$ and that H is pure in the single momentum P. Then in the Schrödinger momentum (P-diagonal) representation

(3.9) $P \equiv pI \Rightarrow H = H(p)I; \quad Q \equiv i\hbar\frac{\partial}{\partial p}; \quad \frac{i}{\hbar}(HQ - QH) = \frac{dH}{dp}I$

so H is a scalar function of scalar p. It follows that (3.7/8) simplifies to a first order DE with operator coefficients

(3.10)
$$\frac{1}{2}\left(\frac{dH}{dp}\frac{i}{\hbar}(p\Theta - \Theta p) + \frac{i}{\hbar}(p\Theta - \Theta p)\frac{dH}{dp}\right) = \frac{i}{\hbar}(H\Theta - \Theta H)$$
$$\Rightarrow -\left(H - \frac{1}{2}\frac{dH}{dp}p\right)\Theta + \Theta\left(H - \frac{1}{2}p\frac{dH}{dp}\right) - \frac{1}{2}\frac{dH}{dp}\Theta p + \frac{1}{2}p\Theta\frac{dH}{dp} = O$$

Now suppose that

(3.11) $\frac{dH}{dp}\Theta p = p\Theta\frac{dH}{dp}$

so only the first two terms remain on the LHS of (3.10). Then, given that Θ is arbitrary, we may equate the pre and post coefficients of Θ, in (3.10), to any operator V that commutes with Θ. The simplest way to specify V is to assume that, like Θ, it is pure in Q. Thus

(3.12) $H - \frac{1}{2}p\frac{dH}{dp} = V(Q) \Rightarrow H = \frac{1}{2m}p^2 + V(Q); \quad dV/dp = O$

where m is a constant scalar. We see that (3.11) holds, both sides being equal to $mp\Theta p$. Because H is no longer pure in P we need to replace dH/dp by $\partial H/\partial p$ when writing (3.10).

Notice that, independently of the above argument, the substitution

(3.13) $H = ap; \quad a$ constant

also satisfies (3.10). Because (3.7) is linear in H we may add the solutions (3.12/13) to give

$$(3.14a)\ H = \frac{1}{2m} p^2 + ap + V(Q)$$

which, in symbols independent of representation, is expressed as

$$(3.14b)\ H = \frac{1}{2m} P^2 + aP + V(Q)$$

Thus, we conclude, that if Θ is arbitrary then constraint 1 requires that H is a polynomial in P of order at most 2.

The solutions of (3.7) can be further generalized on account of its linearity. Because the $\underline{P}$ mutually commute we may add solutions of the type (3.14) for each component of momentum

$$(3.15)\quad H = \sum_{j=1}^{n_c} \left(\frac{1}{2m_j} P_j^2 + a_j P_j \right) + V(\underline{Q})\text{; the } m_j, a_j \text{ and } k \text{ are scalars}$$

where V can be pure in *all* the operators $\underline{Q}$. When $a_j = 0\ \forall\ j$ (3.15) is recognizable as the form of the Newtonian energy of the system of particles in P; (with the proviso that the m_j associated with a given particle are equal).

The process of generalization may be taken further. It can be shown, by substitution, that the Hamiltonian

$$(3.16)\quad H = AP_l; \quad A \equiv b^k P_k,$$

where the Hermitian operator A is linear in the $\underline{P}$ with coefficients that are c-numbers, satisfies (3.7) when Θ is arbitrary. But this is true for all l. We conclude, by linear superposition of expressions (3.15/16) multiplied by coefficients that commute with Θ, that the *most general polynomial form* allowed is

$$(3.17)\quad H \equiv K\{C^{uv}, P_u, P_v\} + \{E^j, P_j\} + U; \quad C^{uv} = C^{vu}; \quad u, v, j = 1,2,\ldots n_c;\ \text{see (2.35a)}$$

The operators C^{uv}, E^j, U, by hypothesis, do *not* depend on the $\underline{P}$. They should be observables and so have real spectra. The algebraic symmetry of the RHS of (3.17) then ensures that the spectrum of H is also real. They must also commute with the $\underline{Q}$ because, if they do not so commute, they must depend on the $\underline{P}$ [2]; this is contrary to hypothesis. The condition $C^{uv} = C^{vu}$ does not compromise generality.

If the coefficients mutually commute, as well as commuting with the $\underline{Q}$ then, they can be represented either by real constants or by Hermitian operator functions of the coordinate operators $\underline{Q}$. They may also be represented by diagonal matrices of such elements. If they do not mutually commute then they must be represented by Hermitian matrices of such elements (as in the Dirac equation [1]). In what follows we assume that the coefficients C^{uv}, E^j, U are mutually commuting functions of the coordinates $\underline{Q}$.

This simplification can be justified by the following argument. If our aim is to deduce the rules of CM from axioms of QM then the velocity operators, like the coordinate operators, must have a classical character; that is, they must have *continuous spectra*. Given (3.17) the velocity operator of the k^{th} coordinate is

$$\text{(3.17a)}\quad \dot{Q}^k \equiv \lfloor H, Q^k \rfloor = 2\{C^{uk}, P_u\} + E^k; \quad \text{see (3.17) and (2.17)}$$

It follows that, if any of the C^{uk} or the E^k are non-scalar matrices then, the spectrum of $\dot{Q}^k$ is *not* continuous; thus, to reach our declared aim, the C^{uv}, E^j must be scalars.

There is a complication. If the Heisenberg representation of velocity fluctuates very fast, e.g., if the mass involved is high, then the measurement of velocity produces an *average*. It can be shown [22] that, if the velocity operator is represented by a matrix and the system is non-degenerate then, the measurement, i.e., the average, is representated by a diagonal matrix. But the diagonal still consists of discrete eigenvalues unless the matrix is scalar.

A further simplification of (3.17) is possible. Terms like $P_u C^{uv} P_v$ can be put into the form either $P_u P_v C^{uv} +$ terms linear in the $\underline{P}$ or $C^{uv} P_u P_v +$ (terms linear in the $\underline{P}$)$^+$. The linear terms can then be subsumed into the terms $\{E^j, P_j\} + U$ at (3.17) to give

$$\text{(3.18)}\quad H \equiv \mathrm{K}\{G^{uv}, P_u P_v\} + \{F^j, P_j\} + V; \quad G^{uv} = G^{vu}; \quad u, v, j = 1,2,\ldots n_c; \ \text{see (2.35c)}$$

as the most general form allowed by constraint 1 when θ is arbitrary. The scalar constant K can be chosen to have the physical dimension of (mass)$^{-1}$. Thus, if H has the dimension of energy then , the G^{uv} have no dimensions. The F^j have the dimension of velocity and V has the dimension of energy.

We now prove, for completeness, that (3.18) satisfies (2.26). Given that

$$\text{(3.18a)}\quad AQ^u = Q^u A; \quad A \text{ pure in the } \underline{Q}$$

we find

$$\lfloor P_j P_k , A \rfloor = P_j A_{,k} + A_{,j} P_k ; \quad A_{,j} \equiv \lfloor P_{,j} , A \rfloor$$

$$\text{(3.18b)} \Rightarrow \lfloor H, A \rfloor = \frac{K}{2}[G^{uv}(P_u A_{,v} + A_{,u} P_v) + (P_u A_{,v} + A_{,u} P_v) G^{uv}] + F^v A_{,v} ; \text{ see (3.18)}$$

$$= \{ H^{:v}, A_{,v} \}$$

By setting $A \equiv \Theta$ in (3.18b) we have (2.26).

At Section 2.9 we show that, given the Hamiltonian operators (2.35a/c), possible ambiguities of the quantization (2.26) are illusory. But we have also shown at (3.18) that the form (2.35c) is the most general allowed by constraint 1. Another general result stems from (3.18). Suppose that A is any polynomial of the form (3.18); also suppose that B is pure in the $\underline{Q}$. Then

$$\text{(3.18c)} \; \lfloor A, B \rfloor = \{ A^{:j}, B_{,j} \}$$

Notice that A can be any derivation of H (with respect to the $\underline{P}$ and/ or the $\underline{Q}$) or any symmetric linear superposition of same with coefficients that are pure in the $\underline{Q}$.

The reverse quantization of (3.18) is the classical Hamiltonian

$$\text{(3.19a)} \; H \equiv K g^{uv}(\underline{q}) p_u p_v + f^j(\underline{q}) p_j + v(\underline{q}) ; \text{ see (2.35b)}$$

where the same scalar functions g^{uv}, f^j, v appear as coefficients in the Q-diagonal representation of the operator (3.18). That is, in the Q-diagonal representation,

$$\text{(3.19b)} \; \begin{array}{l} P_j \equiv -i\hbar \dfrac{\partial}{\partial q^j} ; \quad Q^j \equiv q^j I ; \\ G^{uv} \equiv g^{uv}(\underline{q}) I ; \quad g^{uv} = g^{vu} ; \quad F^j \equiv f^j(\underline{q}) I ; \quad V \equiv v(\underline{q}) I \end{array}$$

Most classical Hamiltonians are quadratic in the momenta; the quadratic form is an expression of Newton's laws in classical analytical mechanics. This is a further justification of the name 'momenta' for the conjugates $\underline{P}$ of the coordinate operators $\underline{Q}$. It follows that the result (3.18) is of high importance, being the first success of CT.

3.3 Hamilton's Equations In Operators Are Valid In QM

By inspection of (2.17/21/22) we see that

$$\text{(3.20a)} \; \dot{P}_j = -H_{,j} ; \quad \dot{Q}^k = H^{:k}$$

These equations are the quantum analogue of the classical Hamilton's equations

(3.20b) $\dot{p}_j = -\frac{\partial H}{\partial q^j}; \quad \dot{q}^k = \frac{\partial H}{\partial p_k}$

Indeed (3.20b) can be regarded as the reverse quantization of (3.20a). But keep in mind that the axioms of quantization require that P is flat and the coordinates are Cartesian; so the QM Hamilton's equations (3.20a) are restricted to flat coordinates. Their classical counterparts $(\underline{p}, \underline{q})$ at (3.20b) are allowed to be curvilinear.

3.4. Constraints 1 And 2 Combined- The Theta Equation

The work of Section 3.1 suggests that H and θ should satisfy, consecutively, as many constraints as possible. It follows that it is desirable that H and θ should satisfy, at the least, constraint 2 as well as constraint 1; we expect θ to be restricted thereby. Constraint 2 is

(2.27) $Z_2 \equiv \left\{ \lfloor H, H^{\cdot j} \rfloor \Theta_{,j} \right\} + \left\{ H^{\cdot j}, H^{\cdot k}, \Theta_{,j,k} \right\} = \lfloor H, H, \Theta \rfloor$

By combining this with (2.26/3.7) we can remove all explicit reference to the $H^{\cdot j}$ and to H. We obtain thereby an operator equation involving only the derivations of Θ. In the Q-diagonal Schrodinger representation this reduces to a fourth order PDE satisfied by θ. The PDE contains no reference to either the f^u or v. It allows us, in principle, to calculate functions $\theta(\underline{q})$ that satisfy both constraints 1 and 2 given the functions g^{uv}; see (3.18/19b).

Commute (2.26/3.7) with H to produce

(3.21) $\lfloor H, Z_1 \rfloor = \lfloor H, H, \Theta \rfloor$

Subtract (3.21) from (2.27) to get

$$\begin{aligned} & Z_2 = \lfloor H, Z_1 \rfloor \Rightarrow \left\{ H^{\cdot j}, H^{\cdot k}, \Theta_{,j,k} \right\} + \left\{ \lfloor H, H^{\cdot j} \rfloor \Theta_{,j} \right\} = \left\lfloor H, \left\{ H^{\cdot j}, \Theta_{,j} \right\} \right\rfloor \\ \text{(3.22a)} & \Rightarrow \left\{ H^{\cdot j}, H^{\cdot k}, \Theta_{,j,k} \right\} = \left\lfloor H, \left\{ H^{\cdot j}, \Theta_{,j} \right\} \right\rfloor - \left\{ \lfloor H, H^{\cdot j} \rfloor \Theta_{,j} \right\} = \left\{ H^{\cdot j}, \lfloor H, \Theta_{,j} \rfloor \right\}; \text{ see (3.18c)} \\ & = \left\{ H^{\cdot j}, \left\{ H^{\cdot k}, \Theta_{,j,k} \right\} \right\} \end{aligned}$$

Written in full the equation of the first and last terms is

$$\text{(3.22b)} \quad \begin{aligned} & \left\{ H^{\cdot j}, H^{\cdot k}, \Theta_{,j,k} \right\} = \left\{ H^{\cdot j}, \left\{ H^{\cdot k}, \Theta_{,j,k} \right\} \right\} \\ & \Rightarrow \frac{1}{12} (H^{\cdot j} H^{\cdot k} \Theta_{,j,k} + \Theta_{,j,k} H^{\cdot j} H^{\cdot k} - 2 H^{\cdot j} \Theta_{,j,k} H^{\cdot k}) = O; \quad \Theta_{,j,k} = \Theta_{,k,j} \end{aligned} ; \text{ see (2.23)}$$

or

(3.22c) $\frac{1}{12}\left[H^{:j},\left[H^{:k},\Theta_{,j,k}\right]\right]=O; \quad \Theta_{,j,k}=\Theta_{,k,j}$

Because

(3.23) $\left[H^{:l},A\right]=2\mathrm{K}G^{ul}A_{,u}; \quad AQ^{u}=Q^{u}A$; see (3.18)

the result (3.22c) is identical to

(3.24) $\frac{\hbar\mathrm{K}}{6i}\left[H^{:j},G^{uk}\Theta_{,j,k,u}\right]=O$

giving, upon further application of (3.23),

(3.25) $-\frac{\hbar^{2}\mathrm{K}^{2}}{3}G^{vj}(G^{uk}\Theta_{,j,k,u})_{,v}=O; \quad j,k,u,v=1,2,...n_{c}$; Operator Theta Equation

The numerical factor $-\hbar^{2}\mathrm{K}^{2}/3$ can, of course, be cancelled; we retain this factor at (3.25) because the operator on the LHS is the imbalance across (5.22a/b). In the Q-diagonal Schrödinger representation

(3.26a) $G^{uv}\equiv g^{uv}(\underline{q})I; \quad g^{uv}=g^{vu}; \quad \Theta=\theta(\underline{q})I$

and (3.25) reduces to the PDE

(3.26b) $g^{vj}(g^{uk}\theta_{,jku})_{,v}=0; \quad j,k,u,v=1,2,...n_{c}$; Scalar Theta Equation

Notice that the theta equation does not contain the functions f^{j} and v. Further the theta equation is not a classical approximation; it is a purely QM result. It can be derived by substituting the quadratic operator H, which derives from the first quantization, namely constraint 1, into the second quantization, namely, constraint 2.

That θ satisfies the PDE (3.26b) raises an immediate issue: Quadratic operator H derives from constraint 1 on the assumption that θ is arbitrary; but it cannot be truly arbitrary if it satisfies (3.26b). At most it is the general solution of (3.26b) given a particular n_{c}-space C. So is the quadratic form of the operator H valid? Yes! Solutions of (3.26b) must be subject to complicated boundary conditions; thus θ is sufficiently arbitrary for the quadratic solution of constraint 1 to follow.

The scalar theta equation (3.26b) can be regarded as a field equation for theta (subject to possible modification by higher constraints at levels 3 and above). Because the

g^{lm} the f^l and v inform the Hamiltonian they are all candidates for θ. We thus have three versions of (3.26b) that are putative field equations for the g^{lm} the f^l and v:

(3.27a/b/c) $$g^{vj}(g^{uk}g^{lm}{}_{,jku})_{,v} = 0; \quad g^{vj}(g^{uk}f^{l}{}_{,jku})_{,v} = 0; \quad g^{vj}(g^{uk}v_{,jku})_{,v} = 0$$

Recall that, according to the quantization axioms, these equations are true only in a flat space using flat (Cartesian) coordinates.

3.5 An Alternative Way To Calculate Z_2 Leads To Questions About Consistency

Condition (3.22a/b) ensures that constraints 1 and 2 shall be consistent. But, when constraint 1 holds, there is an alternative way to calculate Z_2; this, ostensibly, leads to a second consistency condition. Suppose that a scalar f is a function of the $\underline{p}$ and the $\underline{q}$

(3.28) $$f \equiv f(\underline{p},\underline{q}); \quad \underline{p} \equiv \underline{p}(t); \quad \underline{q} \equiv \underline{q}(t)$$

then

(3.29) $$\dot{f} = \frac{\partial f}{\partial p_j}\dot{p}_j + \frac{\partial f}{\partial q^k}\dot{q}^k = -\frac{\partial f}{\partial p_j}\frac{\partial H}{\partial q^j} + \frac{\partial f}{\partial q^k}\frac{\partial H}{\partial p_k}$$

Quantizing according to the rules of Section 2

(3.30) $$\lfloor H, F \rfloor = \{F^{:j}, -H_{,j}\} + \{F_{,k}, H^{:k}\} \quad f \to F$$

Now define

(3.31) $$F \equiv Z_1$$

This definition is consistent with (3.28), that a scalar $f \equiv z_1$ is a function of the $\underline{p}$ and the $\underline{q}$, because

(3.32) $$\frac{\partial H(\underline{p},\underline{q})}{\partial p_l}\frac{\partial \theta(\underline{q})}{\partial q^l} \to \{H^{:l}, \Theta_{,l}\} = Z_1; \text{ see (2.26)}$$

Therefore

(3.33) $$\begin{aligned} Z_2 &= \lfloor H, Z_1 \rfloor = \{Z_1^{:j}, -H_{,j}\} + \{Z_{1,k}, H^{:k}\} \\ &= \{\{H^{:j}, \Theta_{,j}\}_{,k}, H^{:k}\} - \{\{H^{:l}, \Theta_{,l}\}^{:j}, H_{,j}\} \end{aligned}$$

So the second consistency condition equates two expressions for Z_2 (see (2.27))

(3.34a) $\{\lfloor H, H^{:j}\rfloor, \Theta_{,j}\} + \{H^{:j}, H^{:k}, \Theta_{,j,k}\} = \{\{H^{:j}, \Theta_{,j}\}_{,k}, H^{:k}\} - \{\{H^{:l}, \Theta_{,l}\}^{:j}, H_{,j}\}$

or

(3.34b) $\lfloor H, H, \Theta \rfloor = \lfloor H, \{H^{:j}, \Theta_{,j}\}\rfloor = \{\{H^{:j}, \Theta_{,j}\}_{,k}, H^{:k}\} - \{\{H^{:l}, \Theta_{,l}\}^{:j}, H_{,j}\}$; see (2.26/27)

Written in full this is

$$
\begin{aligned}
& -\frac{1}{\hbar^2}\left[H(H\Theta - \Theta H) - (H\Theta - \Theta H)H\right] \\
(3.34c) \quad & = \frac{i}{\hbar}\left[H\left(H^{:j}\Theta_{,j} + \Theta_{,j}H^{:j}\right) - \left(H^{:j}\Theta_{,j} + \Theta_{,j}H^{:j}\right)H\right] \qquad ;\ \text{given constraint 1} \\
& = \frac{1}{2}\begin{bmatrix} H^{:k}\left(H^{:j}\Theta_{,j} + \Theta_{,j}H^{:j}\right)_{,k} + \left(H^{:j}\Theta_{,j} + \Theta_{,j}H^{:j}\right)_{,k}H^{:k} \\ - H_{,j}\left(H^{:l}\Theta_{,l} + \Theta_{,l}H^{:l}\right)^{:j} - \left(H^{:l}\Theta_{,l} + \Theta_{,l}H^{:l}\right)^{:j}H_{,j} \end{bmatrix}
\end{aligned}
$$

We now treat condition (3.34) cautiously by substituting special cases! Try the linear Hamiltonian form

(3.35)
$$
\begin{aligned}
& G^{uv} \equiv 0 \Rightarrow H \equiv \{F^u(\underline{Q}), P_u\} + V(\underline{Q}); \\
& \Rightarrow H^{:j} = F^j; \quad H^{:j:l} = O; \quad H_{,k} = \{F^u, P_u\}_{,k} + V_{,k}
\end{aligned}
$$

Then the LHS of (3.34b) is

(3.36a) $\lfloor H, \{H^{:j}, \Theta_{,j}\}\rfloor = \lfloor H, F^j\Theta_{,j}\rfloor = F^u(F^j\Theta_{,j})_{,u}$

and the RHS of (3.34b) is

(3.36b) $\{\{H^{:j}, \Theta_{,j}\}_{,k}, H^{:k}\} - \{\{H^{:l}, \Theta_{,l}\}^{:j}, H_{,j}\} = \{(F^j\Theta_{,j})_{,k}, F^k\} = F^k(F^j\Theta_{,j})_{,k}$

This result shows that (3.34) is an identity given at least one Hamiltonian form that satisfies constraint 1 for arbitrary θ.

Now try the quadratic form in which the coefficients G^{uv}, F^u, V are constant scalars. The operator H and its derivations mutually commute and, in addition, we note that

(3.37) $$\lfloor H, A(\underline{Q}) \rfloor = \{ H^{:j}, A_{,j} \} \quad H^{:j} = 2\mathrm{K} G^{uj} P_u + F^j; \\ H_{,k} = O; \quad H^{:j}_{,k} = O; \quad H^{:j:l} = 2\mathrm{K} G^{jl}; \quad G^{uv} = G^{vu}$$

It follows that the LHS of (3.34b) is

(3.38a) $$\lfloor H, \{ H^{:j}, \Theta_{,j} \} \rfloor = \{ H^{:j}, \lfloor H, \Theta_{,j} \rfloor \} = \{ H^{:j}, \{ H^{:k}, \Theta_{,j,k} \} \}$$

and the RHS of (3.34b) is

(3.38b) $$\{ \{ H^{:j}, \Theta_{,j} \}_{,k}, H^{:k} \} - \{ \{ H^{:l}, \Theta_{,l} \}^{:j}, H_{,j} \} = \{ \{ H^{:j}, \Theta_{,j,k} \} H^{:k} \}$$

So again, because j and k are dummy and $\Theta_{,j,k} = \Theta_{,k,j}$, (3.34) is an identity.

These special cases tempt one to suppose that (3.34) is an identity given only that constraints 1 and 2 hold. If that is so then the second condition adds nothing. It is, however, not necessarily so. The second consistency condition might require, for example, that the g^{lm} the f^l and v are, *a fortiori*, the candidates for θ. This would be the case if it could be proved, for example, that

(3.39) $$\{ G^{vj} (G^{uk} G^{lm}_{,j,k,u})_{,v}, P_l P_m \} + A \{ G^{vj} (G^{uk} F^l_{,j,k,u})_{,v} P_l \} + B G^{vj} (G^{uk} V_{,j,k,u})_{,v} = O$$

where A and B are non-null and either scalar or pure in the $\underline{Q}$. This relation is then possible only if

(3.40) $$G^{vj} (G^{uk} G^{lm}{}_{,jku})_{,v} = 0; \quad G^{vj} (G^{uk} F^l{}_{,jku})_{,v} = 0; \quad G^{vj} (G^{uk} V_{,jku})_{,v} = 0$$

which, in the Q-diagonal representation, reduces to (3.27a/b/c). Unfortunately, the algebra is too sticky for me!

3.6 Some Free Particle Hamiltonians That Satisfy All Or Many Of The Constraints

We now consider some Hamiltonians of particles that are 'free' in the sense that their acceleration operators are null. For example

(3.41) $$H \equiv C P_1; \quad n_c = 1$$

where C is an Hermitian matrix of constant c-numbers and the single momentum is conjugate to the single coordinate Q_1. We find

(3.42a) $$\dot{Q}_1 = H^{:1} = C; \quad \ddot{Q}_1 = O \Rightarrow \overset{n}{Q}_1 = O \text{ for } n > 1; \quad n = 1,2,....$$

(3.42b) $\dot{\Theta}(Q_1) = C\Theta_{,1}; \quad \ddot{\Theta}(Q_1) = C^2\Theta_{,1,1} \Rightarrow \overset{n}{\Theta}(Q_1) = C^n\Theta_{,1,....,1}$

From now on, when dealing with the case $n_p = n_c = 1$, use the shorthand

(3.42c) $P_1 \equiv P; \quad Q_1 \equiv Q; \quad \Theta_{,1,....,1} \equiv \Theta_{1....1}$

By virtue of (3.42a) only a single term in Z_n survives

(3.43) $Z_n = \{C, C, ..., C, \Theta_{11....1}\} = C^n\Theta_{11....1}; \quad n = 1,2,...$; see (3.42a)

It follows that, because Θ satisfies (3.42b),

(3.43a) $Z_n = \overset{n}{\Theta}$

and the Hamiltonian (3.41) satisfies *all* the constraints for all Θ.

This argument may be extended to the Hamiltonian

(3.44) $H \equiv C^j P_j$

where the $\underline{C}$ mutually commute, commute with the $\underline{P}$ and the $\underline{Q}$, are constant and Hermitian. We find

(3.45a) $\dot{Q}^k = H^{:k} = C^k; \quad \ddot{Q}^k = O \Rightarrow \overset{n}{Q}^k = O$ for $n > 1; \quad n = 1,2,....$

(3.45b) $\dot{\Theta} = C^j\Theta_{,j}; \quad \ddot{\Theta} = C^j C^k \Theta_{,j,k} \Rightarrow \overset{n}{\Theta} = C^{j_1}C^{j_2}....C^{j_n}\Theta_{,j_1,j_2,....,j_n}$

By virtue of (3.45a) only a single term in Z_n survives

(3.46) $Z_n = \{C^{j_1}, C^{j_2}, ..., C^{j_n}, \Theta_{,j_1,j_2,....,j_n}\} = C^{j_1}C^{j_2}....C^{j_n}\Theta_{,j_1,j_2,....,j_n}; \quad n = 1,2,...$

It follows that, because Θ satisfies (3.45b), (3.43a) holds and *all* the constraints are satisfied by the Hamiltonian (3.44) for all Θ.

Consider the Hamiltonian

(3.47) $H \equiv \frac{P^2}{2}; \quad n_c = 1$

This represents the Newtonian kinetic energy of a single free particle, of unit mass, moving in a 1-space. We find (with the convention (3.42c))

$$(3.48)\quad \dot{Q} = H^{:1} = P;\quad \ddot{Q} = O \Rightarrow \overset{n}{Q} = O \text{ for } n > 1;\quad n = 1,2,....$$

The n^{th} constraint is therefore

$$(3.49)\quad \overset{n}{\Theta} = \left\lfloor \frac{P^2}{2}, \frac{P^2}{2}, ..., \frac{P^2}{2}, \Theta \right\rfloor = \{P, P, ...P, \Theta_{1....1}\} = Z_n$$

Constraint 1 is an identity

$$(3.50)\quad \left\lfloor \frac{P^2}{2}, \Theta \right\rfloor = \tfrac{1}{2}(P\Theta_1 + \Theta_1 P);\quad \{P, \Theta_1\} = \tfrac{1}{2}(P\Theta_1 + \Theta_1 P)$$

as we should expect (because H is quadratic). Constraint 2 is

$$(3.51a)\quad \left\lfloor \frac{P^2}{2}, \frac{P^2}{2}, \Theta \right\rfloor = \{P, P, \Theta_{11}\}$$

which reduces to

$$(3.51b)\quad -(P^2\Theta_{11} + \Theta_{11}P^2)/12 + P\Theta_{11}P/6 = O \Rightarrow \Theta_{1111} = O \Rightarrow \theta_{,1111} = 0$$

The only function of the coordinates that characterises the simple system, represented by (3.47), is $\theta = q$; this choice certainly satisfies (3.51b). No function of the coordinate operators appears in the Hamiltonian.

We now show that all higher constraints are satisfied if (3.51b) holds. Constraint 3 is

$$(3.52a)\quad \left\lfloor \frac{P^2}{2}, \frac{P^2}{2}, \frac{P^2}{2}, \Theta \right\rfloor = \{P, P, P, \Theta_{111}\}$$

which reduces to

$$(3.52b)\quad -P^3\Theta_{111} - \Theta_{111}P^3 + P\Theta_{111}P^2 + P^2\Theta_{111}P = O$$

Commute (3.51b) with P and then post multiply by $12P$ to get

$$(3.53a)\quad -P^3\Theta_{111} - P\Theta_{111}P^2 + 2P^2\Theta_{111}P = O$$

Commute (3.51b) with P and then pre multiply by $12P$ to get

(3.53b) $-P^2\Theta_{111}P-\Theta_{111}P^3+2P\Theta_{111}P^2=O$

Add (3.53a/b) to get (9.52b). Thus, if constraint 2 holds (see (3.51)) then, so does constraint 3. Now if (3.51) holds

(3.54) $\Theta_{11111}=O; \quad \Theta_{111111}=O; \quad \Theta_{1....1}=O$

by successive commutations of (3.51b) with P. Constraint 4 is

(3.55) $\left[\frac{P^2}{2},\frac{P^2}{2},\frac{P^2}{2},\frac{P^2}{2},\Theta\right]=\{P,P,P,P,\Theta_{1111}\}$

The RHS of this equation is linear in Θ_{1111} (with pre and post coefficients which are multiples of powers of P) and therefore vanishes. But the LHS is also linear in Θ_{1111}; for example $\left[\frac{P^2}{2},\frac{P^2}{2},\Theta\right]$ is linear in Θ_{11} and $\left[\frac{P^2}{2},\frac{P^2}{2},\frac{P^2}{2},\Theta\right]$ is linear in Θ_{111}. So the LHS also vanishes and constraint 4 is satisfied. By a similar argument, according to (3.54), all the higher constraints are satisfied. Thus, if Θ satisfies (3.51b) then the Hamiltonian (3.47) satisfies *all* the constraints.

Consider the Hamiltonian

(3.56) $H\equiv \mathrm{K} g^{jk}P_jP_k=\mathrm{K}\sum_{j=1}^{n_c}\varepsilon_{jj}P_j^2; \quad g^{jk}\equiv\varepsilon_{jj}\delta_{jk}; \quad \varepsilon_{jj}\equiv\pm 1$

This archetype is of some importance to physics; it belongs to the family (3.18). We find

(3.57a) $\dot{Q}^k=H^{:k}=2\varepsilon_{kk}P^k\Rightarrow\ddot{Q}^k=O; \quad \overset{n}{Q^k}=O$ for $n>1; \quad n=2,3,....$

where the expression for $\dot{Q}^k$ is not summed over k and, for simplicity, we set

(3.57b) $\mathrm{K}=1$

Because the Hamiltonian (3.56) is quadratic constraint 1 is satisfied; likewise constraint 2 is satisfied providing that the theta equation (3.25/26) holds

(3.57c) $g^{vj}g^{uk}\Theta_{,j,k,u,v}=\sum_{j,k=1}^{n_c}\varepsilon_{jj}\varepsilon_{kk}\Theta_{,j,j,k,k}=O\Rightarrow\sum_{j,k=1}^{n_c}\varepsilon_{jj}\varepsilon_{kk}\theta_{,jjkk}=0$

Again there is no function of the coordinates contained in the Hamiltonian; and, again, we choose the coordinates themselves as plausible candidates for θ. With this choice (3.57c) is satisfied.

Constraint 3 is satisfied if (see (2.28)

(3.58a) $$\left[\sum_{j=1}^{n_c}\varepsilon_{jj}P_j^2,\sum_{k=1}^{n_c}\varepsilon_{kk}P_k^2,\sum_{l=1}^{n_c}\varepsilon_{ll}P_l^2,\Theta\right]=\sum_{j,k,l=1}^{n_c}\left\{2\varepsilon_{jj}P_j,2\varepsilon_{kk}P_k,2\varepsilon_{ll}P_l,\Theta_{,j,k,l}\right\}$$

Expanding, keeping in mind that $\Theta_{,j,k,l}$ is independent of the order of its suffices and that those suffices are dummy, the LHS is

(3.58b) $$\begin{aligned}&\sum_{j,k,l}\varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\left(P_j[P_k(P_l\widetilde{\Theta}+\widetilde{\Theta}P_l)+(P_l\widetilde{\Theta}+\widetilde{\Theta}P_l)P_k]+[P_k(P_l\widetilde{\Theta}+\widetilde{\Theta}P_l)+(P_l\widetilde{\Theta}+\widetilde{\Theta}P_l)P_k]P_j\right)\\&=\sum_{j,k,l}\varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}(P_jP_kP_l\widetilde{\Theta}+\widetilde{\Theta}P_jP_kP_l+3P_j\widetilde{\Theta}P_kP_l+3P_jP_k\widetilde{\Theta}P_l)\end{aligned}$$

and the RHS is

(3.58c) $$8\sum_{j,k,l}\varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\left\{P_j,P_k,P_l,\widetilde{\Theta}\right\}=2\sum_{j,k,l}\varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}(P_jP_kP_l\widetilde{\Theta}+\widetilde{\Theta}P_jP_kP_l+P_j\widetilde{\Theta}P_kP_l+P_jP_k\widetilde{\Theta}P_l)$$

where

(3.58d) $$\widetilde{\Theta}\equiv\Theta_{,j,k,l}$$

Collecting terms (3.58a) becomes

(3.59) $$\sum_{j,k,l}\varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\left(-P_jP_kP_l\widetilde{\Theta}-\widetilde{\Theta}P_jP_kP_l+P_j\widetilde{\Theta}P_kP_l+P_jP_k\widetilde{\Theta}P_l\right)=O$$

Now we deduce, above, that

(3.22b) $$\frac{1}{12}(H^{:j}H^{:k}\Theta_{,j,k}+\Theta_{,j,k}H^{:j}H^{:k}-2H^{:j}\Theta_{,j,k}H^{:k})=O$$

which is the general condition that (3.18) shall satisfy constraint 2 given that Θ is arbitrary. Substituting the special case (3.56/57) we have

$$\text{(3.60)}\quad \sum_{j,k} \varepsilon_{jj}\varepsilon_{kk}\left(P_j P_k \Theta_{,j,k} + \Theta_{,j,k} P_j P_k - 2P_j \Theta_{,j,k} P_k\right) = O$$

as another way of writing (3.57c). Commute (3.60) with P_l and then post multiply by $-P_l\varepsilon_{ll}$ to get

$$\text{(3.61a)}\quad -\sum_{j,k,l} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\left(P_l P_j P_k \widetilde{\Theta} + P_l \widetilde{\Theta} P_j P_k - 2P_l P_j \widetilde{\Theta} P_k\right) = O$$

after summing over l. Commute (3.60) with P_l and then pre multiply by $-P_l\varepsilon_{ll}$ to get

$$\text{(3.61b)}\quad -\sum_{j,k,l} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\left(P_j P_k \widetilde{\Theta} P_l + \widetilde{\Theta} P_j P_k P_l - 2P_j \widetilde{\Theta} P_k P_l\right) = O$$

after summing over l. Add (3.61a/b) to get (3.59). Thus, if Θ satisfies (3.57c) then, the Hamiltonian (3.56) satisfies constraints 2 and 3.

Let us try to push this argument further. Constraint 4 is

$$\text{(3.62a)}\quad \begin{aligned} &\left[\sum_{j=1}^{n_c} \varepsilon_{jj} P_j^2, \sum_{k=1}^{n_c} \varepsilon_{kk} P_k^2, \sum_{l=1}^{n_c} \varepsilon_{ll} P_l^2, \sum_{m=1}^{n_c} \varepsilon_{mm} P_m^2, \Theta\right] \\ &= \sum_{j,k,l=1}^{n_c} \left\{2\varepsilon_{jj} P_j, 2\varepsilon_{kk} P_k, 2\varepsilon_{ll} P_l, 2\varepsilon_{mm} P_m, \Theta_{,j,k,l}\right\} \end{aligned}$$

Expanding, the LHS is

$$\text{(3.62b)}\quad \begin{aligned} &\sum_{j,k,l} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\varepsilon_{mm}[P_m(P_j P_k P_l \widetilde{\Theta} + \widetilde{\Theta} P_j P_k P_l + 3P_j \widetilde{\Theta} P_k P_l + 3P_j P_k \widetilde{\Theta} P_l) \\ &+ (P_j P_k P_l \widetilde{\Theta} + \widetilde{\Theta} P_j P_k P_l + 3P_j \widetilde{\Theta} P_k P_l + 3P_j P_k \widetilde{\Theta} P_l) P_m] \\ &= \sum_{j,k,l} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\varepsilon_{mm}(P_j P_k P_l P_m \widetilde{\Theta} + \widetilde{\Theta} P_j P_k P_l P_m \\ &+ 4P_m \widetilde{\Theta} P_j P_k P_l + 6P_m P_j \widetilde{\Theta} P_k P_l + 4P_m P_j P_k \widetilde{\Theta} P_l) \end{aligned}$$

and the RHS is

$$\text{(3.62c)}\quad \begin{aligned} &16\sum_{j,k,l} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\varepsilon_{mm}\left\{P_j, P_k, P_l, P_m, \widetilde{\Theta}\right\} \\ &= \frac{16}{5}\sum_{j,k,l} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\varepsilon_{mm}(P_j P_k P_l P_m \widetilde{\Theta} + \widetilde{\Theta} P_j P_k P_l P_m + P_j \widetilde{\Theta} P_k P_l P_m + P_j P_k \widetilde{\Theta} P_l P_m + P_j P_k P_l \widetilde{\Theta} P_m) \end{aligned}$$

where

(3.62d) $\tilde{\Theta} \equiv \Theta_{,j,k,l,m}$

Collecting terms (3.62a) is

$$\text{(3.63)}\quad \frac{1}{5}\sum_{j,k,l} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\varepsilon_{mm}(-11P_jP_kP_lP_m\tilde{\Theta} - 11\tilde{\Theta}P_jP_kP_lP_m + 4P_m\tilde{\Theta}P_jP_kP_l + 14P_mP_j\tilde{\Theta}P_kP_l + 4P_mP_jP_k\tilde{\Theta}P_l) = O$$

Commute (3.61a/b) with P_m and then pre and post multiply the results by $P_m\varepsilon_{mm}$ to get

$$\text{(3.64a)}\ -\sum_{j,k,l,m} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\varepsilon_{mm}\left(P_mP_lP_jP_k\tilde{\Theta} + P_mP_l\tilde{\Theta}P_jP_k - 2P_mP_lP_j\tilde{\Theta}P_k\right) = O$$

$$\text{(3.64b)}\ -\sum_{j,k,l,m} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\varepsilon_{mm}\left(P_lP_jP_k\tilde{\Theta}P_m + P_l\tilde{\Theta}P_jP_kP_m - 2P_lP_j\tilde{\Theta}P_kP_m\right) = O$$

$$\text{(3.64c)}\ -\sum_{j,k,l,m} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\varepsilon_{mm}\left(P_jP_k\tilde{\Theta}P_lP_m + \tilde{\Theta}P_jP_kP_lP_m - 2P_j\tilde{\Theta}P_kP_lP_m\right) = O$$

$$\text{(3.64d)}\ -\sum_{j,k,l,m} \varepsilon_{jj}\varepsilon_{kk}\varepsilon_{ll}\varepsilon_{mm}\left(P_mP_jP_k\tilde{\Theta}P_l + P_m\tilde{\Theta}P_jP_kP_l - 2P_mP_j\tilde{\Theta}P_kP_l\right) = O$$

after summing over m. Now the four equations (3.64) are derived from (3.60) via (3.61); and (3.60) is an expression of the condition that Θ must satisfy in order that the Hamiltonian (3.56) shall satisfy constraint 2 and, as we have proved, constraint 3. So, if a linear combination of the equations (3.64)

(3.65) $w\,(3.64\text{a}) + x\,(3.64\text{b}) + y\,(3.64\text{c}) + z\,(3.64\text{d})$

exists that is identical to (3.63) then, constraint 4 is also satisfied. Equation (3.65) is identical to (3.63) if there are four numbers w, x, y, z that satisfy the five linear equations

$$\text{(3.66a)}\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11 \\ -11 \\ 14 \\ 4 \\ 4 \end{pmatrix}$$

These equations are found to be consistent; they have the joint solution

(3.66b) $w = -11;\quad x = 4;\quad y = -11;\quad z = -22$

It follows that (3.56) satisfies constraints 1 to 4.

Clearly the method employed here, to prove that the satisfaction of constraints 3 and 4 follows from that of constraint 2, is cumbersome and cannot be extended easily to higher levels. Nevertheless it seems likely that the Hamiltonian (3.56) does indeed satisfy all the constraints given (3.57c).

3.7 An Hypothesis About The Error Operators $\overset{n}{\Theta} - Z_n$

The operator

(3.67) $E_s \equiv \overset{s}{\Theta} - Z_s; \quad s = 1,2,...;$ see (2.26/…./29)

represents the error made by expressing $\overset{s}{\Theta}$ as Z_s. The operator (theta) equation (3.25) is the result of the satisfaction of constraints 1 and 2. It can be expressed as

(3.68) $E_2 = O; \; E_1 = O;$ see (2.26/…./29, 3.2, 5.11)

It is clear that there exist higher level theta equations of the form

(3.69) $E_k = O; \quad k = n, n-1,...,1; \quad n > 2$

In general we are concerned, in what follows, with the non-null error operators that remain when all the error operators below a certain level are null. Thus

(3.70) $E_{n+1} \neq O; \quad E_k = O; \quad k = n, n-1,...,1; \quad n > 1$

Now suppose that the constraints (3.4) are not necessarily satisfied. We see that

(3.71) $\Xi_1(t) \equiv \Theta(t) - \Theta - \sum_{j=1}^{\infty} \frac{Z_j t^j}{j!} = \sum_{j=1}^{\infty} \frac{E_j t^j}{j!};$ see (3.1)

is the total error in $\Theta(t)$ made by expressing the $\overset{s}{\Theta}$ by the Z_s. In particular, if (3.70) holds then,

(3.72) $\Xi_{n+1}(t) \equiv \sum_{j=n+1}^{\infty} \frac{E_j t^j}{j!}; \quad n > 1$

According to the calculation leading to (3.25), when $E_1 = O$

(3.73) $E_2 = -\frac{\hbar^2 K^2}{3} G^{vj} (G^{uk} \Theta_{,j,k,u})_{,v}; \quad j,k,u,v = 1,2,...n_c$

or, in the Q-diagonal Schrodinger representation,

(3.74) $E_2 = -\frac{\hbar^2 K^2}{3} g^{vj} (g^{uk} \theta_{,jku})_{,v} I$

The hypothesis referred to in the title of this section is that, when it happens that

(3.75a) $E_n \neq O; \quad n > 2; \quad E_k = O; \quad k = n-1, n-2, ..., 1$

and H is quadratic in the $\underline{P}$, the error operator has a particularly simple form

(3.75b) $E_n \propto (\hbar K)^n \vartheta_n I; \quad n > 2; \quad E_k = O; \quad k = n-1, n-2, ..., 1$

where, in the Q-diagonal Schrodinger representation, the operator ϑ_n is a scalar function of the $\underline{q}$. This function ϑ_n is presumed to depend in a simple way on the g^{uv}, their coordinate derivatives and the coordinate derivatives of θ. When H is quadratic in the $\underline{P}$ the highest derivative expressions in the g^{uv} and/ or θ, in ϑ_n, are expected to be of order $2n$; then ϑ_n has the physical dimensions of $\theta \times (\text{coordinate})^{-2n}$. The constant of proportionality in (3.75b) is not expected to be big; see (3.74). The hypothesis is clearly true when $n = 2$.

The series on the RHS of (3.72) always converges rapidly if

(3.76) $\frac{|E_{j+1}| t}{j+1} >> |E_j|; \quad j = 3,4,....$; assuming that the theta equation holds

where $|E_j|$ denotes the upper magnitude of the *finite* bounds of the operator E_j. When either bound is infinite (3.76) may be true only when $|E_j|$ denotes the actual magnitude (in the Q-diagonal Schrodinger representation) of E_j at any point in *some finite* region of P. Notice that, if the hypothesis (3.75b) is true (in the Q-diagonal Schrodinger representation) and ϑ_n has the physical dimensions stated then, $E_j t^j$ has the dimensions of θ. Also successive terms in the series (3.76) have the dimensionless ratio

(3.77) $\frac{tE_{j+1}}{(j+1)E_j} = \frac{\hbar K t \vartheta_{j+1}}{(j+1)\vartheta_j}; \quad j = 3,4,....$; assuming that the theta equation holds

The series (3.72) certainly converges if the sequence $\vartheta_{j+1}/\vartheta_j$ is absolutely convergent. Further, if the ratio (3.77) is always very small then, the higher constraints ($n > 2$) are unimportant; that is, the RHS of (3.72) approximates its first term and, for most circumstances, the first term will be physically small. That is

(3.78) $\Xi_3(t) \approx \dfrac{E_3 t^3}{3!}$; assuming that the theta equation holds

Given that the hypothesis (3.75b) holds it is, at the least, plausible that, for laboratory measures of physical quantities, the ratio (3.77) is always very small. In such circumstances *the theta equation (3.25) and quadratic Hamiltonians are sufficient for CM.* For example suppose

(3.79) $\hbar \approx 10^{-34}\ mksu;\quad \mathsf{K} = 1000\ kg^{-1};\quad t = 3\times 10^{13}\ sec;\quad j = 3 \Rightarrow \dfrac{\hbar \mathsf{K} t}{(j+1)} \approx 7.5\times 10^{-20}\ m^2$

That is K is chosen to be the reciprocal of a gram mass and t is chosen to be approximately a million years. With these choices, for the ratio (3.77) to be as high as 10^{-5}, we require $\vartheta_{j+1}/\vartheta_j$ to be slightly higher than $10^{14}\ m^{-2}$. If this is our 'threshold' then, to make the higher constraints important, θ needs to change appreciably over a distance as small as $\sqrt{10^{-14}} = 10^{-7}\ m$ which is a tenth of a micron. Keeping in mind that θ is a characteristic of the system large changes, over such a small distance, is probably at the limit of usefulness of classical analysis.

4. The Coordinate Space C

4.1 Identification Of The Matrix Potentials (Coefficients G^{uv}) And The Time t

The reverse quantization of (3.18) is the classical Hamiltonian

(3.19a) $H \equiv \mathsf{K} g^{uv}(\underline{q}) p_u p_v + f^j(\underline{q}) p_j + v(\underline{q})$; see (2.35b)

where the same scalar functions g^{uv}, f^j, v appear as coefficients in the Q-diagonal representation of the operator (3.18). Now suppose, for simplicity, that

(4.1) $f^j = 0\ \forall\ j;\quad v = 0$

then (3.19a) reduces to

(4.2) $H \equiv \mathsf{K} g^{uv}(\underline{q}) p_u p_v$

and the Hamilton's equations (3.20b) give, after eliminating the $\underline{p}$,

(4.3) $$\ddot{q}^j + \Gamma^j_{kl}\dot{q}^k\dot{q}^l = 0; \quad \Gamma^l_{ij} \equiv \frac{1}{2}g^{lk}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right)$$

The relations (4.3) have the form of the tensor equations for a geodesic, with parameter t, in a Riemannian n_c-space having metric tensor g^{uv}. The geodesic distance is proportional to the parameter

(4.4) $$s \propto t$$

Further, if (3.20b) is the reverse quantization of (3.20a) then, t, at (4.4), is the quantum time (unaffected by the reverse quantization). Note that (4.2/3/4) amount to a classical approximation because they are based on (3.19a).

4.2 The Coordinate Space C And The Representative Point Q

The functions $g^{uv} = g^{vu}$ appear to be elements of a fundamental tensor in a Riemannian n_c-space. Denote this space by C. The coordinates $\underline{q}$ of the particles in P are those of a single representative point Q in C. The classical equations of motion of this point derive from Hamilton's equations operating on (3.19a). The motions of each of the particles in P are thus determined by the time variation of the corresponding subset of $\underline{q}$ along the path of Q. We must *define* the space C to be flat; if it is otherwise there will be topological inconsistencies in using the same set of coordinates $\underline{q}$ in both P and C.

If we accept the above interpretation of the $g^{uv} = g^{vu}$ then the space C is Riemannian metrical and flat. But if P contains only one particle then its coordinates are those of Q and we can say that P ≡ C. So P is Riemannian metrical and flat. This conclusion is consistent with the quantization rules; see Section 2.2 et seq.

Notice that if

(4.5) $$f^j \neq 0 \,\exists\, j \wedge / \vee\, v \neq 0$$

then the path of Q in C is not a geodesic but, presumably, the other interpretations are unaltered.

4.3 The Theta Equation, As It Stands, Is No Use For Explaining Gravity

The equations (3.27a), derived from (3.25) by assigning $\theta \equiv g^{lm}$, are, apparently, of no use in explaining Einsteinian gravity. In the first place, (3.25) is not then a tensor equation. In the second place, if the fundamental tensor g^{lm} is to describe Einsteinian gravity then C must be *curved*. We see this, at once, by considering a single particle in Minkowskian P with Galilean coordinates. In this case the properties of C may be made identical to those of P if we choose the g^{lm} to be Minkowskian; the four coordinates of Q are, necessarily, Galilean. But that is as far as we can go. By definition C is flat. Neither the actual particle in P nor the representative point in C can behave as if subject to Einsteinian gravity.

4.4 The Curved 'Coordinate Space' C'- Kilmister's Equation

Consider a *curved* Riemannian n_c-space C' with general coordinates $\underline{q}'$. Suppose that C is tangential to C' at common points O' in C' and O in C. Then the coordinates $\underline{q}'$ can be chosen to be locally Cartesian geodesic at pole O' such that

(4.6) $\quad \dot{\underline{q}} = \dot{\underline{q}}'; \quad \dot{x} \equiv \dfrac{dx}{dt}; \quad t$ proportional to the geodesic distance in C

Thus (3.25) can be said to hold in both spaces in a neighbourhood of $O' \equiv O$. Also true in that neighbourhood are the classical equations of motion of Q. But those equations of motion are tensor equations (given the assumptions that the f^j form a vector and v is an invariant) true in all coordinate systems. It follows that we may choose

(4.7) $\quad O \equiv$ Q with flat coordinates $\underline{q}$; $\quad O' \equiv$ Q' with geodesic coordinates $\underline{q}'$

and the equations of motion for Q' will be the same tensor equations expressed with respect to the coordinates $\underline{q}'$. The difference between C and C' is that in the latter space the metric tensor, and therefore the curvature, is unrestricted.

If the theta equation (3.25) holds only at the pole of geodesic coordinates then it is approximated in a neighbourhood of that pole. This idea is consistent with the notion that most quantum events, associated with particles, take place in a small region of space.

The question arises: What tensor equation, defined in C', reduces to (3.27a) at the pole O' of locally Cartesian geodesic coordinates when $\theta = g^{lm}$? This question has been answered by Kilmister (see below). The tensor equation is

(4.8) $\quad g^{ef}(R_{ab;ef} + \tfrac{2}{3}R_{ae}R_{fb}) = 0$; R_{ab} is the covariant Ricci tensor; $a,b,e,f = 1,2,...n_c$

where $(.)_{;j}$ denotes covariant differentiation with respect to the j^{th} coordinate. This equation can be interpreted as governing the curvature of C'. It seems to allow us to make contact with Einsteinian gravity. If so then it applies only to the space between particles.

The question also arises: What is the status of the Kilmister equation? Is it quantum mechanical or is it classical? The theta equation, as we have seen, is quantum mechanical. But the Riemannian character of C' depends on the identification of the functions g^{lm} as the metric in a geodesic equation; and that depends on the reverse quantization (3.19a). So the Riemannian character of C' is a classical approximation. Because it is derived from the theta equation, on the assumption that g^{lm} is the metric of a Riemannian space, we need to regard the Kilmister equation as a classical approximation.

If the Kilmister equation can, in fact, describe gravity then it seems that gravity is an *emergent property* of macrophysical systems.

4.5 A Change Of Notation

In what follows, for simplicity, we drop the primes appended to C', O', Q' and $\underline{q}'$ except where there may be ambiguity. When C' happens to be flat C' and C are identical and we may choose $\underline{q}' \equiv \underline{q}$ and $Q' \equiv Q$. When C' might be curved we emphasize this by using standard notation for the coordinates in C' and a consonant notation for the representative point Q'

(4.9) $\underline{x} \equiv \underline{q}'$; $X \equiv Q'$

Thus, in the new notation: C may be curved and the coordinates $\underline{x}$ of the representative point X in C cannot then be the aggregate of the coordinates $\underline{q}$ of particles in P. But if the $\underline{x}$ are geodesic at X then we may choose

(4.10) $\underline{\dot{q}} = \underline{\dot{x}}$; at X. See (4.9)

5. Derivation Of The Kilmister Equation

5.1 Note On Geodesic And Local Cartesian Canonical Coordinates

Before considering Kilmister's answer to the question: "To which tensor equations, expressed in Cartesian geodesics at a pole X, are the equations (3.27a) equivalent"? it is wise to review certain technical matters!

It can be shown [6] that if a point **A** is sufficiently close to a point **B** then the path defined by the geodesic equation, between **A** and **B**, is unique; in the n_c-space C, n_c such paths from n_c points **B**, terminating at the same point **A**, can be used as coordinate axes within a neighbourhood of their origin **A**. We shall call such coordinates Geodesic with pole **A**.

Kilmister's argument uses special geodesic coordinates known as Canonical Coordinates [7]; these can also be local Cartesian. It is convenient to gather here certain facts about such coordinate systems. Suppose that we express a non-singular transformation from coordinates $\bar{x}^j$ to coordinates x^k

(5.1) $x^a = x^a(\bar{x}^b)$

and, further suppose that the RHS of (5.1) can be expanded as a series about any point **P**

(5.2) $x^a = A^a_b \bar{x}^b + B^a_{bc}\bar{x}^b\bar{x}^c + C^a_{bcd}\bar{x}^b\bar{x}^c\bar{x}^d + ...$; summation convention in force

where the constants B^a_{bc} and C^a_{bcd} are symmetrical in their lower suffices. Note that the x^a have been chosen so that the two sets of coordinates have a common origin **P**. Now consider a neighbourhood of **P** in which

(5.3) $x^a \approx A^a_b \bar{x}^b + B^a_{bc}\bar{x}^b\bar{x}^c + C^a_{bcd}\bar{x}^b\bar{x}^c\bar{x}^d; \quad \bar{x}^a \to o$

Neglecting, for the moment, the second and third order terms the constants A^a_b can be chosen so that the x^a are orthogonal [6] in a neighbourhood of **P**; and, therefore, the x^a can be regarded as Cartesian locally to **P**. The constants B^a_{bc}, being equal in number to the $g^{uv}_{,w}$, can be chosen so that, in the x^a system,

(5.4) $g^{uv}_{,w} = 0$

is satisfied at **P**. This is the primary property of geodesic coordinates [8].

Kilmister assumes that the $\bar{x}^a$ are already orthogonal (local Cartesian) geodesic and he applies a further transformation

(5.5) $x^a \to \bar{x}^a + C^a_{bcd}\bar{x}^b\bar{x}^c\bar{x}^d$

leading to a different set of geodesic coordinates. He then applies a set of conditions on the C^a_{bcd}. Namely, that at **P**,

(5.6a) $\Gamma^a_{bc,d} + \Gamma^a_{cd,b} + \Gamma^a_{db,c} = 0$; $\Gamma^a_{bc,d}$ is a Christoffel symbol of the second kind [8]

these being constraints on the second derivatives $g_{jk,lm}$. The coordinates (5.5) are then said to be canonical [7].

Is this procedure valid? Because Γ^a_{bc} is symmetrical in the suffices bc it follows from (5.6a) that

(5.6b) $\Gamma^a_{cb,d} + \Gamma^a_{dc,b} + \Gamma^a_{bd,c} = 0$

Thus there are as many conditions (5.6) as there are coefficients C^a_{bcd}. So the C^a_{bcd} can always be set to enforce (5.6). But we must still investigate whether or not the conditions (5.6) could constrain the curvature tensor [8]. There are

(5.7) $$n_c + n_c(n_c - 1) + \frac{n_c(n_c-1)(n_c-2)}{3!} = n_c(n_c^2 + 3n_c + 2)/6$$

conditions (5.6) which constrain the

(5.8) $$n_c^2(n_c+1)^2/4$$

second derivatives of the g_{uv}. It follows that there are still

(5.9) $$n_c^2(n_c+1)^2/4 - n_c(n_c^2+3n_c+2)/6 = n_c^2(n_c-1)^2/12$$

degrees of freedom. This is also the number of independent elements of the curvature tensor [8]. So the conditions (5.6) do not constrain the curvature tensor. Kilmister's example is for $n_c = 4$; but the point he makes applies to all n_c.

There is some complexity, not to say confusion, in the terminology used by various authors for the sort of coordinate system that interests us here. [8] calls systems that satisfy (5.4), at a pole, *geodesic*. [6] calls such systems, with coordinate axes defined by geodesics emanating from and in the neighbourhood of a pole, *normal*. [9] calls such systems *Riemannian*; we call them *geodesic*. [9] calls systems for which, at the pole, the metric tensor is diagonal with elements equal to ± 1, *local Cartesians*. [7] calls such systems *natural*; and, of course, natural coordinates can satisfy (5.4). [9] reserves the term *geodesic* for systems where the Levi-Civita connections vanish at a pole; this criterion can thus apply to non-Riemannian spaces, for which a metric is not defined, as well as to Riemannian spaces. [9] reserves the term *normal* for systems defined by a set of parametric surfaces.

5.2 Kilmister's Derivation Of (4.8)

The following argument was communicated by Clive Kilmister in a letter. Passages or expressions in square brackets, thus [], are mine.

" If one does the cubic transformation

(5.10) $x^a \to \bar{x}^a + C^a_{bcd}\bar{x}^b\bar{x}^c\bar{x}^d \quad [n_c = 4]$

on geodesic coordinates you have 80 coefficients C^a_{bcd} and these are just enough to allow you to apply the conditions ([7], Equ. (36.7), p.79)

(5.11) $\Gamma^a_{bc,d} + \Gamma^a_{cd,b} + \Gamma^a_{db,c} = 0$

(which I say produces 'canonical coordinates'- Ibid. p.79). Now in geodesics

(5.12) $R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d}$; [Riemann-Christoffel or curvature tensor]

(in my sign convention).

From (5.11)

(5.13) $-\Gamma^a_{bc,d} = \Gamma^a_{cd,b} + \Gamma^a_{db,c}$

so that (5.12) becomes

(5.14a) $R^a_{bcd} = 2\Gamma^a_{bd,c} + \Gamma^a_{cd,b}$

Interchange b and c

(5.14b) $R^a_{cbd} = \Gamma^a_{bd,c} + 2\Gamma^a_{cd,b}$

These two give

(5.15) $3\Gamma^a_{cd,b} = 2R^a_{cbd} - R^a_{bcd}$

But

(5.16) $-R^a_{bcd} = R^a_{cdb} + R^a_{dbc} = -R^a_{cbd} + R^a_{dbc}$; [[8], Equs. (32.3) and (32.4), p. 51]

Hence

(5.17) $\Gamma^a_{cd,b} = \frac{1}{3}(R^a_{cbd} + R^a_{dbc})$

This is the key to everything.

Since

(5.18a) $g_{ab,c} = g_{pb}\Gamma^p_{ac} + g_{ap}\Gamma^p_{bc}$; [[8], Equ. (20.4) et seq., p.27] we have [at the pole of geodesics]

(5.18b) $g_{ab,cd} = g_{pb}\Gamma^{p}_{ac,d} + g_{ap}\Gamma^{p}_{bc,d}$

and using (5.17) gives

(5.19) $g_{ab,cd} = \frac{1}{3}(R_{cbad} + R_{cabd})$

Hence

(5.20) $g^{cd}g_{ab,cd} = \frac{2}{3}R_{ab}$

The next step may need a little care. If we want to calculate $R_{ab,ef}$ we can say

$$R_{ab,ef} = (R_{ab;e} + \Gamma^{p}_{ae}R_{pb} + \Gamma^{p}_{be}R_{ap})_{,f}$$

(5.21) $= R_{ab;ef} + \Gamma^{p}_{ae,f}R_{pb} + \Gamma^{p}_{be,f}R_{ap} + [\Gamma^{p}_{ae}R_{pb,f} + \Gamma^{p}_{be}R_{ap,f}]$; [see (5.17)]

$$= R_{ab;ef} + \frac{1}{3}(R^{p}_{afe}R_{pb} + R^{p}_{efa}R_{pb} + R^{p}_{bfe}R_{ap} + R^{p}_{efb}R_{ap})$$

after some cancelling [and because $\Gamma^{p}_{ae} = 0$ at the pole of geodesics]. The first and third terms in the bracket, on the RHS of (5.21), vanish, because R^{p}_{afe} is skew, and the second and fourth terms are equal. So

(5.22a) $g^{ef}R_{ab,ef} = [\frac{3}{2}g^{ef}g^{cd}g_{ab,cdef} =]\, g^{ef}(R_{ab;ef} + \frac{2}{3}R_{ae}R_{fb}) = 0$; [see (5.20)]

[because

(5.22b) $g^{ef}g^{cd}g_{ab,cdef} = 0 \Leftrightarrow g^{vj}g^{uk}g^{lm}_{,jkuv} = 0$]

I know I've been using g_{ab} when you prefer g^{ab} but it is all easily converted.

How splendid canonical coordinates are!"

The last of the equations (5.22a)

(4.8) $g^{ef}(R_{ab;ef} + \frac{2}{3}R_{ae}R_{fb}) = 0$; Kilmister Equation

is clearly a tensor equation, the LHS being composed of sums and products of tensors, and so is the required result.

6. Some Classical Mechanics

6.1 Gravity In Empty Space

By assuming that there is but a single particle in P ($n_p = 1$) and choosing P to be Minkowskian ($n_d = 4$) we have

(6.1) $n_c = 4$

If C is flat and

(6.2) $\underline{x} = \underline{q}$

we can require $C \equiv P$. We can, however, make contact with Einsteinian gravity by further assuming that C is, in general, curved and that, only as the curvature evanesces, is $C \to P$ possible.

The curvature of C is governed by the Kilmister equation which states that the Kilmister tensor vanishes

(4.8) $K_{ab} \equiv g^{ef}(R_{ab;ef} + \frac{2}{3}R_{ae}R_{fb}) = 0; \quad a,b,e,f = 1,2,...n_c$

where R_{ab} is the covariant Ricci tensor. We say that (4.8) can have to do with Einsteinian gravity because, when

(6.3) $f^j = 0; \quad v = 0; \quad n_p = 1; \quad n_d = 4 \Rightarrow n_c = 1$,

the representative point at X moves along a geodesic of C and the particle in P moves so that

(6.4) $\dot{\underline{x}} = \dot{\underline{q}}$

where the coordinates $\underline{x}$ in C are geodesic with pole X. When there is curvature neither X nor the particle in P can move so that $\dot{\underline{x}} = \dot{\underline{q}}$ is constant. Notice that solutions of

(6.5) $R_{ab} = 0$ in C

are always solutions of (4.8) but not vice versa.

If we take the original classical model literally, the space between the point particles in P is empty. It is reasonable to assume, therefore, that, for the purposes of theory, there is no distributed *matter* in C. This idea is supported by the fact that, given suitable dimensionality, signature and metric, (6.5) is Einstein's original law of gravity in the empty space between distributions of matter.

But we may be obliged to accept that physical space-time contains a *vacuum energy*; and, even though C is a mathematical artefact, given (6.3) we can assign suitable properties to C. Further, there seems to be no bar to approximating a large number of particles in P to a *continuous fluid*.

6.2 The Significance Of $n_c = 4$

The dimensionality $n_c = 4$ has a special significance in relation to the laws (6.5) and (4.8). The tensor equation (6.5) expands to

(6.6) $n_c(n_c + 1)/2$

unique PDEs. But (6.5) is the contraction of the tensor equation

(6.7) $R^d_{abc} = 0$

where R^d_{abc} is the Riemann-Christoffel or curvature tensor. The condition (6.7) *defines* the space as flat [8], [9]; and it comprises

(6.8) $n_c^2(n_c^2 - 1)/12$

unique PDEs [7]. We deduce that solutions of the tensor equation (6.5) cannot imply curvature if

(6.9) $n_c(n_c + 1)/2 \geq n_c^2(n_c^2 - 1)/12 \Rightarrow n_c < 4$

because, under condition (6.9), (6.5) will imply (6.7). So, given (6.5), curvature is only possible if

(6.10) $n_c \geq 4$

The same is true for the solutions of (4.8) since (4.8) also comprises $n_c(n_c + 1)/2$ PDEs. When $n_c = 4$, given that there is only one time-like coordinate, the remaining three must be space-like. So, if C is curved then, the minimum dimension allowed for P is four and the geometry of P is then Minkowskian.

6.3 Some properties of (4.8) In Relation To GR

The material of the following section, equs. (6.3) to (6.17), is better set out in the notes [25]. In particular [25] makes clear that (4.8) is entirely geometrical; the Kilmister equation describes the evolution of the metric tensor given four sets of geometrical boundary conditions. If we wish to understand the implications of this solution for matter-energy-stress we must substitute the resulting metric tensor into Einstein's equation (6.11).

We now confirm the assumptions of Section 6.1 so that the Kilmister equation can make contact with GR. As we remark above, in the classical model, there is no matter between the point-particles in P ; and we assume, for the purpose of theory, that there is no distributed matter in C.

Einstein's later law, for gravity in the presence of a continuous distribution of mass-energy-momentum-stress, allows us to investigate this idea. The law is

$$(6.11)\quad G_{ab} + \Lambda g_{ab} + \chi T_{ab} = 0; \quad G_{v}^{u} \equiv R_{v}^{u} - \tfrac{1}{2} R \delta_{v}^{u}; \text{ [3]}$$
$$\chi \equiv 8\pi \mathrm{G} / c^4 = 2.076 \times 10^{-48}\, cm^{-1} gm^{-1} sec^{2}$$

where Λ is the cosmological constant, G is Newton's gravitational constant and T_{ab} is the mass-energy-momentum-stress tensor.

A word about the origin of (6.11). The classical conservation laws of mechanics can be expressed by saying that the divergence of T_{ab} vanishes

$$(6.12)\quad T_{b;a}^{a} = 0$$

Einstein proposed that there should be a geometric tensor that corresponds to T_{ab}; thus the curvature of space-time would be related to the matter content. This tensor, like T_{ab}, would have a zero divergence. The geometric tensor G_{ab} has this property; and Einstein further proposed that these two tensors, the mechanical T_{ab} and the geometric G_{ab}, should be proportional

$$(6.13)\quad G_{ab} + \chi T_{ab} = 0; \quad G_{b;a}^{a} = 0$$

He later added the Λ term on cosmological grounds; and, later still, he came to regard this addition as a blunder. The value given for the constant χ is determined by the requirement that, under conditions of weak gravity and low speed, (6.13) must agree with Newtonian mechanics; see Poisson's equation [10].

If

$$(6.14)\quad T_{ab} = 0$$

that is, the space-time is truly empty then,

(6.15) $G_{ab} = -\Lambda g_{ab} \Rightarrow R_{ab} = \Lambda g_{ab}$.

Substitute this last result into

(4.8) $g^{ef}(R_{ab;ef} + \frac{2}{3}R_{ae}R_{fb}) = 0$; Kilmister Equation

and we get

(6.16) $\Lambda = 0 \Rightarrow R_{ab} = 0$; see (6.5)

Suppose, however, that the space-time is not truly empty; that is, there is no matter between the particles but, there is energy. We then have

(6.17) $G_{ab} = -\Lambda g_{ab} - \chi T_{ab} \Rightarrow R^u_v \equiv G^u_v - \frac{1}{2}G\delta^u_v = -\chi(T^u_v - \frac{1}{2}T\delta^u_v) + \delta^u_v\Lambda$

which can be substituted into (4.8) to give a tensor equation for T_{ab}. If we believe that space-time is truly empty then T_{ab} and Λ vanish and this equation is a null identity; see (6.5/14/16). But, otherwise, (4.8) gives *possible* distributions of energy-momentum-stress. This restriction is not as stringent as might be supposed; because the PDEs (4.8) are of fourth order they are capable of much flexibility.

6.4 Newtonian Approximations

The Kilmister equation is of mind-boggling complexity! Just to set it up, using (say) the spherically symmetric (SS) metric of Schwarzschild [8], we require a computer to do the algebra. The resulting PDEs occupy pages. Formal solution, by computer, fails (so far). Yet the computer does show that the Schwarzschild metric satisfies both (6.5) and (4.8) exactly as it should. As far as I can see we have no recourse but to resort to special cases and approximations.

Suppose that the gravity is weak, so that the space-time is nearly flat, and that matter/ energy moves slowly or not at all. Then we may use quasi-Galilean coordinates and assume that the metric is, approximately, Minkowskian. Thus

(6.18) $$\begin{array}{l} g_{jk} = g^{jk} = 0,\ j \neq k,\ g_{\mu\mu} = -1 + 2\Omega,\ g_{44} = 1 + 2\Omega; \quad \mu = 1,2,3 \\ g^{\mu\mu} = -1 - 2\Omega,\ g^{44} = 1 - 2\Omega; \quad |\Omega| << 1; \quad n_p = 1; \quad n_d = n_c = 4 \end{array}$$ [7], p. 101

where Ω is a dimensionless potential proportional to that of Newton. Since we are aiming at a Newtonian approximation we assume (6.16) at the outset; the Λ term was not envisaged by Newton, is meaningless in the context of his theory and was rejected, ultimately, by Einstein. Note: the expressions (6.18), when compared with those given by Eddington [7], replace Ω with $-\Omega$. The reason for this is that Eddington, in his derivations, refers to

$$(6.18a)\ \Omega = \frac{\mathsf{G}m}{c^2 r}; \quad m > 0$$

as the potential due to a single point particle at the origin; whereas the usual definition is

$$(6.18b)\ \Omega = -\frac{\mathsf{G}m}{c^2 r}$$

so that the Hamiltonian expression for the radial acceleration

$$(6.18c)\ -c^2 \frac{d\Omega}{dr} = -\frac{\mathsf{G}m}{r^2}$$

is negative (i.e., gravitating matter attracts).

We now investigate the T^{ab} dictated by (4.8) in the manner alluded to at (6.17). In the quasi-Newtonian approximation only one element of T^{ab} is non-zero; it represents the energy density in space. Because, by hypothesis, there is no material, between the particles, this energy density is *not* due to matter.

In quasi-Galilean coordinates the stress-free energy tensor is approximated by

$$(6.19)\quad T^{ab} \approx c^2 \rho \frac{dx^a}{ds}\frac{dx^b}{ds} \quad \text{[7], p. 102}$$

where ρ is the proper mass-density equivalent of the supposed energy density $c^2\rho$ (an invariant). Under the conditions cited it is legitimate to make the following further approximation. In the space between particles

$$(6.20)\quad T^{44} = c^2\rho; \quad T^{ab} = 0; \quad T^a_b = T^{au} g_{ub} \Rightarrow T^4_4 = c^2\rho(1+2\Omega) = T; \quad T^a_b = 0; \quad (ab) \neq (44)$$

Therefore

$$(6.21a)\ \begin{array}{l} R^4_4 = -\frac{\chi}{2}c^2\rho(1+2\Omega); \quad R^\mu_\mu = \frac{\chi}{2}c^2\rho(1+2\Omega); \\ R^a_b = 0; \quad a \neq b; \quad \mu = 1,2,3; \quad a,b = 1,2,....4 \end{array} ; \text{ see (6.17)}$$

From (6.18) we get

(6.21b) $R_{ab} = R_a^u g_{ub} \Rightarrow R_{\mu\mu} = -\frac{\chi}{2}c^2\rho; \quad R_{44} = -\frac{\chi}{2}c^2\rho(1+4\Omega)$

neglecting Ω^2 compared to unity. We shall later treat $\chi c^2\rho/2$ as a first order small quantity. Therefore (6.21b) can be simplified to

(6.21c) $\begin{aligned} &R_{\mu\mu} \approx -\frac{\chi}{2}c^2\rho; \quad R_{44} \approx -\frac{\chi}{2}c^2\rho; \\ &R_{ab} = 0; \quad a \neq b; \quad a,b = 1,2,....4 \end{aligned}$

It can be shown [7, p. 102] that, to the same degree of approximation as at (6.18),

(6.22a) $R_{ab} = 0, a \neq b, R_{aa} = \square^2\Omega$; $\square^2 \equiv -\nabla^2 + \frac{\partial^2}{\partial(x^4)^2}$; x^4/c is coordinate time

and, if Ω does not involve the time, then

(6.22b) $R_{ab} = 0, a \neq b, R_{aa} = -\nabla^2\Omega$

Result (6.21c) combined with (6.22b) gives

(6.23) $-R_{aa} = \nabla^2\Omega = \frac{\chi}{2}c^2\rho = \frac{4\pi G}{c^2}\rho$

which is a version of Poisson's equation [10]. In this case, given the conditions for the Newtonian approximation, (6.23) relates the putative energy density ρ to the potential Ω. Now suppose that we substitute (6.22) into (4.8) instituting (6.18), that is, the same level of approximation that leads to (6.22). We have, with quasi-Galilean coordinates,

(6.24) $g^{ef}R_{aa;ef} \approx g^{ef}R_{aa,ef} \approx \square^2(\square^2\Omega); \quad g^{ef}R_{ab;ef} \to 0, a \neq b; \quad g^{ef}R_{ae}R_{fb} \to 0$

Thus

(6.25a) $\square^2(\square^2\Omega) = 0 \Rightarrow \square\rho = 0$; see (6.21c/22/4.8)

and, if Ω and ρ do not involve the time, then

(6.25b) $\nabla^2(\nabla^2\Omega) = 0 \Rightarrow \nabla^2\rho = 0$

We see, at once, that Ω satisfies versions of the theta equation. Given that $C \equiv P$ is Minkowskian and the coordinates are Galilean then the theta equation

(3.26b) $g^{vj}(g^{uk}\theta_{,jku})_{,v} = 0; \quad j,k,u,v = 1,2,...n_c$; Scalar Theta Equation,

with $\theta \equiv \Omega$, becomes (6.25a). Likewise, with $\theta \equiv \Omega$, if $\mathrm{C} \equiv \mathrm{P}$ is Euclidean and the coordinates are Cartesian we get (6.25b). Further the result (6.25a) shows that the energy density propagates according to the wave equation; but, if the energy density is stationary, then the result (6.25b) shows that it satisfies Laplace equation.

There is another way of looking at the first result (6.25b). The QM Hamiltonian for a single Newtonian particle in E3, acted on by a scalar gravitational potential $v(\underline{q})$, is

$$(6.26) \quad H \equiv \frac{1}{2m}\sum_{j=1}^{3} P_j^2 + V(\underline{Q}); \quad \underline{Q} \equiv \underline{q}I; \quad V(\underline{Q}) \equiv v(\underline{q})I$$

where m is the particle mass. Here the Hamiltonian operator is quadratic, the space is Euclidean, the coordinates are Cartesian and v is a function characteristic of the system which, therefore, must be a candidate for θ. Comparing (6.26) with the general case (3.18) we see that the metric is

$$(6.26a) \; g_{jk} = \frac{\delta_{jk}}{2m\mathrm{K}}; \quad n_c = n_d = 3 \text{ because, by hypothesis, } n_p = 1$$

proportional to the Euclidean metric. The theta equation for v is

$$(6.27) \quad 4m^2\mathrm{K}^2\nabla^2(\nabla^2 v) = 0 \Rightarrow \nabla^2(\nabla^2 v) = 0; \text{ see (3.2/ 6.25b)}$$

The relation between v and the dimensionless Ω is

$$(6.28) \quad v = mc^2\Omega$$

That Newtonian approximations of the Kilmister equation produce versions of the theta equation is hardly a surprise; the Kilmister equation derives from the theta equation. But this circumstance provides a check and allows us a limited freedom to calculate.

6.5 Extra Terms In The Newtonian Approximation

The spherically symmetric (SS) solutions of the equations (6.25b) are

$$(6.29) \quad \Omega = \frac{k_1}{r} + k_2 r^2 + k_3 r + k_4$$

and

(6.30) $\rho = \frac{l_1}{r} + l_2$

where $k_1,....k_4, l_1, l_2$ are constants; the symmetry is about the origin. When r is small the terms k_1 / r and l_1 / r dominate (6.29/30); when r is large the term $k_2 r^2$ dominates (6.29) and $\rho = l_2$ is constant. We recognise the k_1 term at (6.29) as corresponding to the Newtonian potential due to a gravitating point mass at the origin (for gravitation $k_1 < 0$). We must not, of course, allow r to become so small that the approximating condition $|\Omega| << 1$ breaks down; see (6.18). We also recognise k_4 as the arbitrary potential floor in the Newtonian theory which, in this case, must be set to zero (to ensure $|\Omega| << 1$). In other words these are the customary terms in the SS solution

(6.31) $\Omega = \frac{k_1}{r} + k_4$

of the usual potential equation for empty space

(6.32) $\nabla^2 \Omega = 0$

But what of the extra terms $k_2 r^2 + k_3 r$ at (6.29)? The usual classical field equations are of second order. The difference between the solutions (6.29) and (6.31) is an illustration of the fact that, because the theta equation is of fourth order instead of second order, its solutions contain two extra terms. To have escaped experimental detection, in this case, the extra terms (those involving k_2 and k_3) must be either zero or very small; if the latter then, perhaps, the corresponding terms are significant only at cosmological distances.

Suppose we regard these extra terms as providing a perturbing radial acceleration

(6.32a) $-c^2 \frac{d(k_2 r^2 + k_3 r)}{dr} = -c^2 (2k_2 r + k_3)$

We see that this perturbation is of quite a different form to the GR term, in the orbital equation, that produces the advancement of the perihelion of Mercury; that term is proportional to $1/r^4$ [10], p. 27 and [8], p. 117.

We could, of course, invoke the principle that the influence of point masses or charges *must* decrease with distance; in which case $k_2 = 0$ and $k_3 = 0$ by fiat. If we invoke this principle on all relevant occasions the new theory may not differ so much from the old. The principle is used, for example, as a boundary condition when calculating the Schwarzschild metric [8], p. 115. It is assumed that, at great distance from a single isolated particle, the space becomes Minkowskian. The principle is also used to reject one of the Bertrand solutions as ‘unphysical’. Bertrand [10], p. 90 found that central orbits, under a power law potential, are closed only if the power is -1 or 2; the latter result was rejected.

We do not invoke the principle here. Rather we allow the extra terms to generate an *energy density* $c^2\rho$ calculable from (6.23)

$$(6.33)\quad \begin{aligned} c^2\rho &= \frac{2}{\chi}\nabla^2\Omega = \frac{2}{\chi}\nabla^2\left(\frac{k_1}{r} + k_2 r^2 + k_3 r + k_4\right) \\ &= \frac{2}{\chi}\left(6k_2 + \frac{2k_3}{r}\right); \quad \nabla^2 \equiv \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \end{aligned}; \text{ see (6.23/29)}$$

If we compare (6.33) with (6.30) we see that some of the constants are connected

$$(6.34)\quad l_2 = -\frac{3}{2}\left(\frac{c^2}{\pi G}\right)k_2; \quad l_1 = -\frac{1}{2}\left(\frac{c^2}{\pi G}\right)k_3$$

If r is large enough then the energy density is appreciably constant at value $12k_2/\chi$; see (6.33). The radial acceleration (along a line to the particle at the origin), of a test particle of infinitesimal mass, is

$$(6.35a)\ -c^2\frac{d\Omega}{dr} = c^2\left(\frac{k_1}{r^2} - 2k_2 r - k_3\right)$$

and, if r is large enough then, this approximates

$$(6.35b)\ -c^2(2k_2 r + k_3) \approx -c^2 2k_2 r; \quad r \to \infty$$

Consider two test particles of infinitesimal mass. Place them at points $x^j_{(1)}$ and $x^k_{(2)}$. Then their Newtonian equations of motion are

$$(6.36)\quad \ddot{x}^\mu = -c^2\Omega_{,\mu}\big|\underline{x} = \underline{x}_{(1)}; \quad \ddot{x}^\nu = -c^2\Omega_{,\nu}\big|\underline{x} = \underline{x}_{(2)}; \quad \mu,\nu = 1,2,3; \text{ see (6.28)}$$

Define the displacement η^μ of one particle with respect to the other by

$$(6.37)\quad x^\mu_{(1)} \equiv x^\mu; \quad x^\mu_{(2)} \equiv x^\mu + \eta^\mu$$

Then, assuming that r is large, (6.29/36) give

$$(6.38)\quad \Omega \approx k_2 r^2; \quad \ddot{\eta}^\mu = -c^2\Omega_{,\nu}\big|\underline{x} = \underline{x}_{(2)} + c^2\Omega_{,\nu}\big|\underline{x} = \underline{x}_{(1)} \approx -2c^2 k_2 \eta^\mu; \quad r \to \infty$$

Thus, at a sufficient distance from the origin, the test particles experience a relative acceleration/ deceleration proportional to their distance apart and *along the line joining them*. By contrast, the terms k_1/r, and $k_3 r$ causes an acceleration/ deceleration of both particles but *in a radial direction.*

Now suppose that there are many gravitating particles at a great distance from the two test particles. Because the first equation (6.38) makes no reference to the origin we have

$$(6.39)\quad \ddot{\eta}^{\mu} \approx -2c^2 K_2 \eta^{\mu}; \quad K_2 \equiv \sum k_2$$

the summation being over the gravitating particles. This last result relies on the additivity assumed for potentials in Newton's theory.

6.6 Dark Energy?

Standard cosmological models [3] predict, without benefit of Λ, that the universe expands, roughly, according to the Hubble law; but, at great distance/ time, the predictions require that the expansion slows. Recently, however (1990s), measurements, using supernovae as standard candles, suggest that, at great distances, the expansion is accelerating. The effect is said to be due to *dark energy.*

To explain this phenomenon some theorists have resurrected the Λ term; but the new Λ has a sign opposite to that of the quantity originally introduced by Einstein which was required to stop the expansion! Other theorists have introduced mysterious scalar fields to explain the extra acceleration. The new theory, however, seems to predict the dark energy effect quite naturally.

We have shown above that, in the Newtonian approximation at least and at great distances from gravitating particles, a constant density of energy is imposed throughout space. This has the effect that two test particles, of infinitesimal mass, experience a relative acceleration/ deceleration *proportional to their distance apart* and *along the line joining them*. This motion is superposed on any other motions (that may be predicted by standard cosmological theories) including the Hubble recession. The constant of proportionality is $-2c^2 K_2$; see (6.39).

Can we estimate a value for K_2? In particular what is its sign? In the present state of the universe the rest energy of the matter (including gas and *dark matter*) dominates. We make the assumption, common to standard cosmologies based on GR, that, over large enough regions, the mean matter-density ρ_0 is constant [3].

We make three additional assumptions in order to apply the above Newtonian analysis to large regions of the universe: a) the curvature is small; b) the velocities of the matter fluid are small compared to c; c) the total energy of the universe is zero. This last is a radical hypothesis; but it gives a definite sign and magnitude to K_2.

As argued above the extra terms, in the potential of a distant gravitating particle, give rise to an energy density $12k_2/\chi$; see (6.33). So, according to the ideas that lead to (6.39), many such particles give rise to a constant total energy density

$$(6.40)\quad c^2\rho = \frac{12}{\chi}\sum k_2 = \frac{12}{\chi}K_2$$

Because both ρ and ρ_0 are appreciably constant assumption c) can be expressed as

$$(6.41)\quad \rho + \rho_0 \to 0 \Rightarrow \rho \approx -\rho_0$$

Given (6.40) this implies

$$(6.42)\quad K_2 \approx -\frac{\chi}{12}c^2\rho_0 < 0; \quad \rho_0 > 0$$

Therefore, according to (6.39), the infinitesimal test particles *recede* from one another. Assign a typical estimate of

$$(6.43)\quad \rho_0 = 8\times 10^{-30}\,gm\ cm^{-3};\ \text{roughly the closure value}$$

and the acceleration of recession per unit of separation is

$$(6.44)\quad -2c^2K_2 = \frac{\chi}{6}c^4\rho_0 = \frac{4}{3}\pi G\rho_0 = 2.236\times 10^{-36}\,sec^{-2};\ \text{see (6.39)}$$

In order to compare (6.39/44) with the Hubble law

$$(6.45)\quad \dot{\eta}^{\mu} \approx H\eta^{\mu} \Rightarrow \ddot{\eta}^{\mu} \approx H^2\eta^{\mu}; \quad H \text{ is the Hubble constant}$$

we compute

$$(6.46)\quad +\sqrt{-2c^2K_2} = +\sqrt{\frac{4}{3}\pi G\rho_0} = 1.495\times 10^{-18}\,sec^{-1} \equiv 14\,km\ sec^{-1}\ per\ 10^6\,ly$$

This value is at the lower bound of modern estimates of the Hubble constant (15 to 30 $km\ sec^{-1}\ per\ 10^6\,ly$). So the effect predicted by the above argument is significant.

The situation studied in the above argument may be summarised as follows. Gravitation is a phenomenon by which (positive) *matter/ energy* tends to clump whereas (negative) *dark energy* tends to spread out. There is a temptation to regard this negative energy as that due to the original *antimatter* which, it is supposed, was an equal companion to the *matter* created by the Big Bang. Hence the above assumption that the total energy of the universe is zero.

6.7 The Need For An Exact Cosmological Investigation

Keep in mind that the assumption (6.41), that $c^2\rho$ (*dark energy-density*?) is the negative of the rest energy of the mean mass-density $c^2\rho_0$, is debatable. Further the analysis is crude. Perhaps the results are misleading. We need a full cosmological investigation which marries the Kilmister equation with (say) a Robertson-Walker model of an homogeneous and isotropic universe [3]. Machine algebra would be a necessity.

A first go at this project is reported in [25] and [26]. The results, although mostly approximate, are very satisfactory; they are summarised, briefly, in the Overview. We may need to recalculate the various cases with an evaluation of Term1 and Term2 in addition to every final calculation of the Kilmister tensor; these would show what the balance of the approximation is in $K_b^a \to 0$. In addition it would be desirable to give analytic expressions for the Ricci and Einstein tensors under the *cosmological metric*; (see the Introduction to [25]).

6.8 Two Particles- Flat C- Electrostatics Or Gravity?

Hitherto, when examining the field equation (4.8) or its flat-space progenitor

(3.25) $g^{vj}(g^{uk}\theta_{,jku})_{,v} = 0$,

we have considered a single particle; thus $n_c = n_d$. This raises a question: how does the new formalism deal with more than one particle?

We consider the simplest problem, involving only (3.25), first! Two particles, in an Euclidean 3-space P, move under a scalar potential according to Newtonian mechanics. We thus have $n_d = 3$ and $n_c = 6$. The appropriate Hamiltonian operator, expressed in the Q-diagonal representation, is

(6.47) $$H \equiv \frac{1}{2}\left(\sum_{j=1}^{3} P_j^2/m_1 + \sum_{j=4}^{6} P_j^2/m_2\right) + \Omega(\underline{Q}); \quad \underline{Q} \equiv \underline{q}I; \quad \Omega(\underline{Q}) \equiv \omega(\underline{q})I; \quad m_1, m_2 > 0$$

where the symbol Ω is now used for the operator that represents the Newtonian potential (energy). Given its position in the Hamiltonian (6.47) the function ω is a candidate for θ. Particle 1 has mass m_1 and Cartesian coordinates q^1, q^2, q^3; and particle 2 has mass m_2 and Cartesian coordinates q^4, q^5, q^6. The metrical coefficients of C are obtained by comparing the expression (6.47) with the general case (3.18)

$$(6.48)\quad g^{uu} = \frac{1}{2m_1\mathrm{K}};\quad u = 1,2,3;\quad g^{vv} = \frac{1}{2m_2\mathrm{K}};\quad v = 4,5,6;\quad g^{jk} = 0;\quad j \neq k$$

where K is a constant with dimensions of (mass) $^{-1}$ that ensures that the metric has no dimensions. The coordinates and the space are flat; thus (3.25) applies to the whole of C. Substituting the metric, with an appropriate choice for θ, (3.25) becomes

$$(6.49a)\quad \sum_{j,k=1}^{6} g^{jj} g^{kk} \theta_{,jjkk} = 0;\quad \theta \equiv \omega$$

In more detail

$$(6.49b)\quad \begin{aligned} &\frac{1}{(2m_1\mathrm{K})^2}\sum_{j,k=1}^{3}\omega_{,jjkk} + \frac{1}{4m_1m_2\mathrm{K}^2}\left(\sum_{j=1,k=4}^{3,6}\omega_{,jjkk} + \sum_{j=4,k=1}^{6,3}\omega_{,jjkk}\right) + \frac{1}{(2m_2\mathrm{K})^2}\sum_{j,k=4}^{6}\omega_{,jjkk} \\ &= \frac{\nabla_1^2\nabla_1^2\omega}{(2m_1\mathrm{K})^2} + \frac{(\nabla_1^2\nabla_2^2 + \nabla_2^2\nabla_1^2\omega)}{4m_1m_2\mathrm{K}^2} + \frac{\nabla_2^2\nabla_2^2\omega}{(2m_2\mathrm{K})^2} = 0 \end{aligned}$$

and then, more succinctly, cancelling through by $1/(4\mathrm{K}^2)$ we get

$$(6.49c)\quad \left(\frac{\nabla_1^2}{m_1} + \frac{\nabla_2^2}{m_2}\right)^2 \omega = 0;\quad m_1, m_2 \neq 0$$

According to Einstein the expression of physical law must be independent of the choice of coordinates. In this case $\omega(\underline{q})$ must be an invariant independent of rotation and translation of the coordinate axes in P. The only such invariants, associated with the classical model, are the distance in P between the two particles

$$(6.50)\quad l \equiv +[(q^1 - q^4)^2 + (q^2 - q^5)^2 + (q^3 - q^6)^2]^{1/2}$$

and functions of same. So ω is a function of l. Therefore

$$(6.51)\quad \nabla_1^2\omega = \nabla_2^2\omega = \left(\frac{d^2}{dl^2} + \frac{2}{l}\frac{d}{dl}\right)\omega$$

and (6.49c) becomes (after division by $(1/m_1 + 1/m_2)^2$)

$$(6.52)\quad \left(\frac{d^2}{dl^2} + \frac{2}{l}\frac{d}{dl}\right)^2 \omega = 0$$

The general solution of this equation is ($c_1,...,c_4$ are constants of integration)

(6.53) $\omega = \frac{c_1}{l} + c_2 l^2 + c_3 l + c_4$; compare with (6.29)

If l is small enough, we may neglect the terms $c_2 l^2 + c_3 l$. Then ω provides an inverse square force (of attraction if $c_1 < 0$ and of repulsion if $c_1 > 0$) along the line joining the particles. In this sense the new field equation (3.25) is successful. The neglected terms are supposed only significant when the distances are cosmological. But, depending on the signs and magnitudes of c_2 and c_3, other behaviours are possible. For example, if $c_2 > 0$ and the kinetic energy is finite then, the particles are (classically) confined, irrespective of the sign of c_1, and l is bounded; (pairs of quarks are confined in an analogous manner).

Strictly speaking, because C is flat, we could say that the result (6.53) does not apply to gravity; rather, we might say, it must apply, if it applies to any force in nature, to electrostatics. But (6.18), which can be described as a Newtonian approximation to a GR metric, leads, via the tensor formalism of GR, to (6.25/29)! Thus Newtonian approximations, as in the conventional physics, require the electrostatic and gravitational potentials to have the same form.

6.9 Two Particles- Flat C With A Different Hamiltonian

According to Newtonian mechanics there is another Hamiltonian operator which might be used, instead of (6.47), in the above argument

(6.54a) $H \equiv \frac{1}{2m'}\left(\sum_{\mu=1}^{3} P'^2_\mu\right) + \Omega(\underline{Q'}); \quad \underline{Q'} \equiv \underline{q'}I; \quad \Omega(\underline{Q'}) \equiv \omega(\underline{q'})I$

which is part of a more general Hamiltonian

(6.54b) $H \equiv \frac{1}{2m_0}P_0^2 + \frac{1}{2m'}\left(\sum_{\mu=1}^{3} P'^2_\mu\right) + \Omega(\underline{Q'}); \quad \underline{Q'} \equiv \underline{q'}I; \quad \Omega(\underline{Q'}) \equiv \omega(\underline{q'})I$

The relationship between these Hamiltonians and that at (6.47) can be summarised as follows:

(6.55) $m_0 \equiv m_1 + m_2 =$ total mass; $\quad \frac{1}{m'} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow m' = \frac{m_1 m_2}{m_0} =$ corrected mass

Define (with a change of notation) centre of gravity operators $Q_{\mu 0}$ and coordinate difference operators Q'_μ

(6.56a) $Q_{\mu 1} \equiv Q^{\mu}; \quad Q_{\mu 2} \equiv Q^{(\mu+3)}; \quad Q_{\mu 0} \equiv \dfrac{m_1 Q_{\mu 1} + m_2 Q_{\mu 2}}{m_0}; \quad Q'_{\mu} \equiv Q_{\mu 2} - Q_{\mu 1}$

and their conjugate momentum operators

(6.56b) $P_{\mu 1} \equiv P_{\mu}; \quad P_{\mu 2} \equiv P_{(\mu+3)}; \quad P_{\mu 0} \equiv P_{\mu 1} + P_{\mu 2}; \quad P'_{\mu} \equiv \dfrac{1}{2}\left(P_{\mu 2} - P_{\mu 1}\right)$

which satisfy as identities (see Section 2)

(6.57) $Q_{\mu 0} P_{\nu 0} - P_{\nu 0} Q_{\mu 0} = i\hbar \delta_{\mu\nu} I; \quad Q'_{\mu} P'_{\nu} - P'_{\nu} Q'_{\mu} = i\hbar \delta_{\mu\nu} I; \quad \mu, \nu = 1,2,3$

Observe that

(6.58) $\dfrac{1}{2m_0} P_0^2 + \dfrac{1}{2m'}\left(\sum_{\mu=1}^{3} P'^2_{\mu}\right) \equiv \dfrac{1}{2}\left(\sum_{j=1}^{3} P_j^2 / m_1 + \sum_{j=4}^{6} P_j^2 / m_2\right)$; see (6.47/54b)

So the only difference between the Hamiltonian operators (6.47) and (6.54b) is the assumption, at (6.54b), that ω depends only on the coordinate differences. Also using (6.54b)

(6.59) $\dot{Q}_{\mu 0} = \left\lfloor H, Q_{\mu 0} \right\rfloor = \dfrac{P_{\mu 1} + P_{\mu 2}}{m_0} \Rightarrow Q'_{\mu} \dot{Q}_{\mu 0} = \dot{Q}_{\mu 0} Q'_{\mu} \Rightarrow \dot{Q}_{\mu 0} \Omega(\underline{Q'}) = \Omega(\underline{Q'}) \dot{Q}_{\mu 0}$

It follows that

(6.60) $\ddot{Q}_{\mu 0} = \left\lfloor H, \dot{Q}_{\mu 0} \right\rfloor = O$

That is, the centre of gravity is in uniform motion providing that ω depends only on the coordinate differences. This fact we know from classical mechanics; but here it has been deduced using only the operator calculus. Further, from (6.54b), using the operator calculus

(6.61) $\dot{Q}'_{\mu} = \left\lfloor H, Q'_{\mu} \right\rfloor = \dfrac{P_{\mu 2}}{m_2} - \dfrac{P_{\mu 1}}{m_1} \Rightarrow \ddot{Q}'_{\mu} = \left\lfloor H, \dot{Q}'_{\mu} \right\rfloor = -\dfrac{1}{m'}\left\lfloor P'_{\mu}, \Omega(\underline{Q'}) \right\rfloor = -\dfrac{1}{m'} \dfrac{\partial \Omega(\underline{Q'})}{\partial Q'_{\mu}}$

which is the operator equation of motion of the coordinate differences. This equation can be deduced using (6.54a/57). So, as foreshadowed at the beginning of this section, we can use (6.54a) instead of either (6.47) or (6.54b) to deduce the equation governing ω providing that we accept that ω depends only on the coordinate differences.

The expression (6.54a) has the appearance of a single particle Hamiltonian; see (6.26). So we must choose $n_p = 1, n_d = 3 \Rightarrow n_c = 3$ and regard the coordinate differences $\underline{q}'$ as the coordinates $\underline{q}$ in an E3. From (6.54a),

(6.62) $$g^{uv} = \frac{\delta_{uv}}{2m'\mathrm{K}}; \quad u,v = 1,2,3;$$ see (6.26a)

Thus (3.25) becomes

(6.63) $$\sum_{j,k=1}^{3} g^{jj} g^{kk} \theta_{,jjkk} = 0; \quad \theta \equiv \omega \Rightarrow \nabla^2(\nabla^2\omega) = 0;$$ see (6.27)

The remainder of the argument parallels that leading to (6.50) and on to (6.53).

6.10 Two Particles- Small Curvature Of C

We now examine the hypothesis that curvature of C characterises gravity. We assert this hypothesis, above, when we attempt to match C to the space-time of GR by choosing $n_d = 4; \quad n_p = 1$. But there is an overriding problem connected with the setting up of gravitational field equations: namely, the choice of a metric suitable to the situation; and this problem persists even with the simplification that the gravitational field is weak and can be represented by a single scalar function of the space-like coordinates ω. The difficulty is that, with more than one particle, the dimension of C is a multiple of that of P . This means that the potential complexity of the metric of C grows rapidly with the number of particles. In general we do not have the prop of a corresponding Hamiltonian (see the previous section) from which to 'read off' a metric for C.

We can, however, make a start by considering what happens when there is no gravity. There is a theorem, concerning flat spaces [9], pp. 59, 82, that allows the metric of C to be of the form

(6.64a) $$ds^2 = \sum_{j=1}^{n_c} \varepsilon_j (dq^j)^2; \quad \varepsilon_j = \pm 1$$

where the $\underline{q}$ are global, orthogonal and flat. The theorem states that we can always set up locally Cartesian coordinates at a given point. If, however, C is flat then a single set of these coordinates can be used everywhere in the space. Thus we have only to choose the pattern ε_j. Suppose that the pattern is known for P; for example, that P is Minkowskian and its coordinates are Galilean. Then it is sufficient that the pattern of the ε_j consists of a repetition of that which pertains in P with one repeat for every particle. If, in addition, we scale the coordinates, by fixed factors, we do not change their orthogonal character; the metric is still diagonal. Thus

(6.64b) $ds^2 = \sum_{j=1}^{n_c} g_{jj}(dq^j)^2; \quad g_{jk} = 0 \;\forall\; j \neq k$

where the g_{jj} are constants.

When the gravity is weak the metric will, we suppose, approximate the forms (6.64). One way to choose the metric of C, in the weak gravity case, is to equate the coordinate accelerations, obtained from an invariant scalar potential ω, with those given by suitably approximated geodesic equations. The resulting equations link the Christoffel symbols of C to the derivatives of ω. We then require that the metric should be one that satisfies these PDEs and demonstrates curvature; naturally, we choose the simplest that will do.

Assume that P contains two particles (n_p=2) and is Minkowskian with Galilean coordinates ($n_d = 4, n_c = 8$). Thus particle 1 has mass m_1, three spacelike coordinates x^1, x^2, x^3 and one timelike coordinate x^4; and particle 2 has mass m_2, three spacelike coordinates x^5, x^6, x^7 and one timelike coordinate x^8. Suppose that all coordinates have the physical dimension of length. Assume that the gravitational field depends only on the instantaneous positions of the particles so that ω is a function of the space-like coordinates alone; this is consistent with the usual Newtonian approach. We have in mind that, without gravity, the repeated Minkowski/ Galilean pattern is then appropriate for the metric of C

(6.65) $g_{jj} = \varepsilon_j \equiv -1,-1,-1,1,-1,-1,-1,1; \quad g_{jk} = 0 \;\forall\; j \neq k;$ see (6.64a)

The approximate equations of motion, generated from ω, are

(6.66) $\ddot{x}^{\mu} \approx -\dfrac{1}{m_1}\omega_{,\mu}; \quad \ddot{x}^{\mu+4} \approx -\dfrac{1}{m_2}\omega_{,(\mu+4)}; \quad \ddot{x}^4 = \ddot{x}^8 = 0; \quad \dot{()} \equiv d()/dt$

where ω has the dimensions of energy. Here Greek suffices have the range $1,2,3$ while Roman suffices have the range $1,2,...,8$. Unless stated otherwise we use this convention for the rest of the calculation. The geodesic equations are, by contrast,

(6.67) $\dfrac{d^2x^l}{ds^2} = -\Gamma^l_{ik}\dfrac{dx^i}{ds}\dfrac{dx^k}{ds}; \quad i,k,l = 1,2,...,8; \quad \Gamma^l_{ij} \equiv \tfrac{1}{2}g^{lk}(g_{ik,j} + g_{jk,i} - g_{ij,k}) = \Gamma^l_{ji}$

Under the approximating assumptions

(6.68) $ds = cdt; \quad \dfrac{dx^4}{ds} \approx 1 \Rightarrow \dot{x}^4 \approx c; \quad \dfrac{dx^8}{ds} \approx 1 \Rightarrow \dot{x}^8 \approx c;$ c is the speed of light

Thus, equating the accelerations (6.67/68),

$$
(6.69)\quad \begin{aligned}
&\frac{1}{m_1}\omega_{,\mu} \approx c^2\Gamma^{\mu}_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds}; \quad \frac{1}{m_2}\omega_{,(\mu+4)} \approx c^2\Gamma^{(\mu+4)}_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds}; \quad \mu = 1,2,3;\\
&c^2\Gamma^4_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} \to 0; \quad c^2\Gamma^8_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} \to 0
\end{aligned}
$$

Equating the coefficients of unity on either side of these equations

$$
(6.70)\quad \begin{aligned}
&\frac{1}{m_1}\omega_{,\mu} = c^2(\Gamma^{\mu}_{44} + \Gamma^{\mu}_{88}); \quad \frac{1}{m_2}\omega_{,(\mu+4)} = c^2(\Gamma^{(\mu+4)}_{44} + \Gamma^{(\mu+4)}_{88}); \quad \mu = 1,2,3\\
&\Gamma^4_{44} + \Gamma^4_{88} = 0; \quad \Gamma^8_{44} + \Gamma^8_{88} = 0
\end{aligned}
$$

The quantities $\Gamma^l_{ik}, \frac{dx^{\mu}}{ds}, \frac{dx^{(\mu+4)}}{ds}, \frac{\omega_{,l}}{c^2 m_1}, \frac{\omega_{,l}}{c^2 m_2}$ are either first order small or zero. It follows that we need not trouble to equate the coefficients of $\dot{x}^i$ or of $\dot{x}^i\dot{x}^j$ at (6.69).

Any choice of the metric is allowed that both satisfies the equations (6.70) and approaches (a possibly scaled version of) the form (6.65) uniformly as $\omega \to 0$. We therefore make a simple choice in the hope that it satisfies all the equations (6.70)! Thus

$$
(6.71)\quad \begin{aligned}
&g_{jk} \equiv g^{jk} = 0,\ j \neq k; \quad g_{\mu\mu} \equiv -Km_1; \quad g_{(\mu+4)(\mu+4)} \equiv -Km_2; \quad g_{44} = g_{88} \equiv 1 + K\frac{\omega}{c^2};\\
&g^{\mu\mu} \approx -\frac{1}{Km_1}; \quad g^{(\mu+4)(\mu+4)} \approx -\frac{1}{Km_2}; \quad g^{44} = g^{88} \approx 1 - K\frac{\omega}{c^2}; \quad 1 >> \left|K\frac{\omega}{c^2}\right|;\\
&m_1, m_2, K > 0
\end{aligned}
$$

from which

$$
(6.72)\quad \begin{aligned}
&\Gamma^{\mu}_{44} = \Gamma^{\mu}_{88} = \frac{1}{2c^2 m_1}\omega_{,\mu}; \quad \Gamma^{(\mu+4)}_{44} = \Gamma^{(\mu+4)}_{88} = \frac{1}{2c^2 m_2}\omega_{,(\mu+4)};\\
&\Gamma^4_{44} = \Gamma^4_{88} = \Gamma^8_{44} = \Gamma^8_{88} = 0
\end{aligned}
$$

Therefore the metric (6.71) satisfies the equations (6.70).

We can calculate the Ricci tensor from

$$
(6.73)\quad R_{ab} \approx \Gamma^s_{as,b} - \Gamma^s_{ab,s}; \quad a,b,s = 1,2,...8;
$$
[8], pp. 50, 52

because the Γ^k_{ij} are either zero or small. We find

(6.74) $$\begin{aligned}\Gamma^s_{as} &= \tfrac{1}{2} g^{sk}(g_{ak,s} + g_{sk,a} - g_{as,k}) = \tfrac{1}{2} g^{aa}(g_{aa,a} + g_{aa,a} - g_{aa,a}) \\ &= \tfrac{1}{2} g^{aa} g_{aa,a} = 0 \ \forall\, a\end{aligned}$$; not summed on a

(6.75) $$\Gamma^s_{ab,s} = \sum_s [\tfrac{1}{2} g^{ss}(g_{as,b} + g_{bs,a} - g_{ab,s})]_{,s} \approx \tfrac{1}{2}\sum_s g^{ss}(g_{as,bs} + g_{bs,as} - g_{ab,ss})$$

From which it follows that the only surviving terms of the type (6.65) are (see (6.72))

(6.76) $$\Gamma^s_{44,s} = \Gamma^s_{88,s} = \frac{1}{2c^2}\sum_\mu \left(\frac{\omega_{,\mu\mu}}{m_1} + \frac{\omega_{,(\mu+4)(\mu+4)}}{m_2}\right) = \frac{1}{2c^2}\left(\frac{\nabla_1^2}{m_1} + \frac{\nabla_2^2}{m_2}\right)\omega;$$

Therefore

(6.77) $$R_{aa} \approx -\frac{\delta_{a4} + \delta_{a8}}{2c^2}\left(\frac{\nabla_1^2}{m_1} + \frac{\nabla_2^2}{m_2}\right)\omega; \quad R_{ab} = 0 \ \forall\, a \neq b;$$ see (6.73/74/76)

With the same approximations (4.8) becomes

(6.78) $$g^{ef}(R_{ab;ef} + \tfrac{2}{3} R_{ae} R_{fb}) \approx g^{ef} R_{ab,ef} \to 0 \Rightarrow \sum_{e=1}^{8} g^{ee} R_{aa,ee} \to 0$$

Substituting (6.71 /77) into (6.78) we have the approximation

(6.79) $$\frac{\delta_{a4} + \delta_{a8}}{2c^2 K}\left(\frac{\nabla_1^2}{m_1} + \frac{\nabla_2^2}{m_2}\right)^2 \omega = 0 \Rightarrow \left(\frac{\nabla_1^2}{m_1} + \frac{\nabla_2^2}{m_2}\right)^2 \omega = 0;$$ see (6.49c)

being the two particle result at (6.49c).

The metric (6.71) suggests that the curvature of C resides in the time-like partition. The question, as to whether the coordinate space is truly flat, can be resolved by calculating the elements of the Riemann-Christoffel tensor

(6.80) $$R^l_{.jnp} \approx \Gamma^l_{jp,n} - \Gamma^l_{jn,p};$$ to first order small quantities [8], p. 50

To demonstrate curvature we do not need to identify all the elements $R^l_{.jnp}$ that may be non-zero; it is sufficient to demonstrate that at least one does not vanish. Potentially non-zero Christoffel symbols are

$$\text{(6.81)}\quad \begin{array}{l} \Gamma^{\mu}_{44} = \Gamma^{\mu}_{88} = \dfrac{1}{2c^2 m_1}\omega_{,\mu}; \quad \Gamma^{(\mu+4)}_{44} = \Gamma^{(\mu+4)}_{88} = \dfrac{1}{2c^2 m_2}\omega_{,(\mu+4)}; \\ \Gamma^{4}_{4\mu} = \Gamma^{8}_{8\mu} = \mathrm{K}\dfrac{\omega_{,\mu}}{2c^2}; \quad \Gamma^{4}_{4(\mu+4)} = \Gamma^{8}_{8(\mu+4)} = \mathrm{K}\dfrac{\omega_{,(\mu+4)}}{2c^2} \end{array} \quad ;\ \text{see (6.72)}$$

whereas

$$\text{(6.82)}\quad \Gamma^{j}_{kl,4} = \Gamma^{j}_{kl,8} = 0$$

giving for example

$$\text{(6.83)}\quad \begin{array}{l} R^{\mu}_{.4\nu 4} \approx \Gamma^{\mu}_{44,\nu} - \Gamma^{\mu}_{4\nu,4} = \dfrac{\omega_{,\mu\nu}}{2c^2 m_1}; \quad R^{\mu}_{.44\nu} \approx \Gamma^{\mu}_{4\nu,4} - \Gamma^{\mu}_{44,\nu} = -\dfrac{\omega_{,sp}}{2c^2 m_1}; \\ R^{\mu}_{.8\nu 8} \approx \Gamma^{\mu}_{88,\nu} - \Gamma^{\mu}_{8\nu,8} = \dfrac{\omega_{,\mu\nu}}{2c^2 m_2}; \quad R^{\mu}_{.88\nu} \approx \Gamma^{\mu}_{8\nu,8} - \Gamma^{\mu}_{88,\nu} = -\dfrac{\omega_{,sp}}{2c^2 m_1}; \\ R^{4}_{.4\mu\nu} \approx \Gamma^{4}_{4\nu,\mu} - \Gamma^{4}_{4\mu,\nu} = 0; \quad R^{8}_{.8\mu\nu} \approx \Gamma^{8}_{8\nu,\mu} - \Gamma^{8}_{8\mu,\nu} = 0 \end{array}$$

Therefore, if any of the $\omega_{,\mu\nu}$ are non-zero, the coordinate space is curved; this is always the case given the solution (6.7). Similar results hold when μ is replaced by $(\mu + 4)$.

6.11 More Than Two Particles- Flat C

In order to discuss the case of $n_p > 2$ we generalise the argument given in Sections 6.8/9. The argument given in Section 6.10 can, almost certainly, be generalised to the case $n_p > 2$; but it introduces extra complications into a topic which is already complicated.

We consider P to be E3 again. Then the simplest choice for the metric of C is

$$\text{(6.84)}\quad g^{jk} = \delta_{jk}; \quad n_c = 3n_p$$

Substituting (6.84) into (3.25)

$$\text{(6.85)}\quad \sum_{j,k}^{n_c}\theta_{,jjkk} = \left(\sum_{\alpha}^{n_p}\nabla^2_{\alpha}\right)^2\theta = 0$$

This equation, although elegant in appearance, does not give a context for θ. To provide such a context we can generalise the Hamiltonian operator (6.47)

$$(6.86) \quad H \equiv \frac{1}{2}\sum_{\alpha}^{n_p}\sum_{\mathrm{j}=1}^{3}\frac{P_{\mathrm{j}\alpha}^2}{m_\alpha} + \Omega(\underline{Q}); \quad \underline{Q} \equiv \underline{q}I; \quad \Omega(\underline{Q}) \equiv \omega(\underline{q})I; \quad m_\alpha > 0$$

where we use the same convention, for the notation of momenta, as pertains in Section 8.2. Greek indices run in the range $1,\ldots,n_p$; Roman Italic indices run in the range $1,\ldots,n_c$; Roman Text indices run in the range 1,2,3. The Hamiltonian (6.86) defines the motion of a collection of Newtonian particles, in E3, under the scalar potential $\omega(\underline{q})$. Then ω is a candidate for θ and comparison of (6.86) with (3.18) gives the metric

$$(6.87) \quad g^{jk} = \frac{\delta_{jk}}{2m_\alpha \mathrm{K}}; \quad j = 1,2,\ldots.n_c; \quad \alpha \equiv ipt(j/3)$$

Substitute (6.87) into (3.25) and we get

$$(6.88) \quad \left(\sum_{\alpha}^{n_p}\frac{\nabla_\alpha^2}{m_\alpha}\right)^2 \theta = 0; \quad \theta \equiv \omega$$

after multiplying through by constants.

As before ω must be a function of the invariants associated with the geometry of the particles in P. These include $n_p(n_p-1)/2$ distances between the particles and various angles etc. Because the angles etc. are functions of the distances we assume that ω is a function only of the distances

$$(6.89) \quad l_{\alpha\beta} = l_{\beta\alpha} \equiv +\left[\sum_{\mathrm{J}}^{3}(q^{\mathrm{j}\alpha} - q^{\mathrm{j}\beta})^2\right]^{1/2}; \quad \alpha \neq \beta = 1,2,\ldots.,n_p$$

where we use the same convention for notation of coordinates as at (6.86). When we move the α^{th} particle we change the distances $l_{\alpha\beta}\ \forall\, \beta \neq \alpha$ given α. Therefore, referring to change of the coordinates of the γ^{th} particle,

$$(6.90) \quad \frac{\partial\omega}{\partial q^{\mathrm{j}\gamma}} = \sum_{\substack{\alpha \\ \alpha\neq\gamma}}^{n_p}\frac{\partial l_{\alpha\gamma}}{\partial q^{\mathrm{j}\gamma}}\frac{\partial\omega}{\partial l_{\alpha\gamma}} \Rightarrow \frac{\partial^2\omega}{\partial(q^{\mathrm{j}\gamma})^2} = \sum_{\substack{\alpha \\ \alpha\neq\gamma}}^{n_p}\left(\frac{\partial^2 l_{\alpha\gamma}}{\partial(q^{\mathrm{j}\gamma})^2}\frac{\partial\omega}{\partial l_{\alpha\gamma}} + \sum_{\substack{\beta \\ \beta\neq\gamma}}^{n_p}\frac{\partial l_{\alpha\gamma}}{\partial q^{\mathrm{j}\gamma}}\frac{\partial l_{\beta\gamma}}{\partial q^{\mathrm{j}\gamma}}\frac{\partial^2\omega}{\partial l_{\alpha\gamma}\partial l_{\beta\gamma}}\right)$$

where (see (6.89))

(6.91) $$\frac{\partial l_{\alpha\beta}}{\partial q^{j\gamma}} = \frac{(q^{j\alpha} - q^{j\beta})(\delta_{\alpha\gamma} - \delta_{\beta\gamma})}{l_{\alpha\beta}}; \quad \frac{\partial^2 l_{\alpha\beta}}{\partial (q^{j\gamma})^2} = \frac{(\delta_{\alpha\gamma} - \delta_{\beta\gamma})^2}{l_{\alpha\beta}}\left(1 - \frac{(q^{j\alpha} - q^{j\beta})^2}{l_{\alpha\beta}^2}\right)$$

and so

(6.92) $$\sum_{j=1}^{3} \frac{\partial^2 l_{\alpha\gamma}}{\partial (q^{j\gamma})^2} = \frac{2(\delta_{\alpha\gamma} - 1)^2}{l_{\alpha\gamma}}; \quad \sum_{j=1}^{3} \frac{\partial l_{\alpha\gamma}}{\partial q^{j\gamma}} \frac{\partial l_{\beta\gamma}}{\partial q^{j\gamma}} = \sum_{j=1}^{3} \frac{(q^{j\alpha} - q^{j\gamma})(q^{j\beta} - q^{j\gamma})(\delta_{\alpha\gamma} - 1)(\delta_{\beta\gamma} - 1)}{l_{\alpha\gamma} l_{\beta\gamma}}$$
$$= \sum_{j=1}^{3} \frac{(q^{j\alpha} - q^{j\gamma})(q^{j\beta} - q^{j\gamma})}{l_{\alpha\gamma} l_{\beta\gamma}}$$

Thus $\dfrac{\partial \omega}{\partial q^{j\gamma}} = \displaystyle\sum_{\substack{\alpha \\ \alpha \neq \gamma}}^{n_p} \frac{\partial l_{\alpha\gamma}}{\partial q^{j\gamma}} \frac{\partial \omega}{\partial l_{\alpha\gamma}} \Rightarrow \frac{\partial^2 \omega}{\partial (q^{j\gamma})^2} = \sum_{\substack{\alpha \\ \alpha \neq \gamma}}^{n_p} \left(\frac{\partial^2 l_{\alpha\gamma}}{\partial (q^{j\gamma})^2} \frac{\partial \omega}{\partial l_{\alpha\gamma}} + \sum_{\substack{\beta \\ \beta \neq \gamma}}^{n_p} \frac{\partial l_{\alpha\gamma}}{\partial q^{j\gamma}} \frac{\partial l_{\beta\gamma}}{\partial q^{j\gamma}} \frac{\partial^2 \omega}{\partial l_{\alpha\gamma} \partial l_{\beta\gamma}} \right)$

(6.93) $$\nabla_\gamma^2 \omega \equiv \sum_{j=1}^{3} \frac{\partial^2 \omega}{\partial (q^{j\gamma})^2} = \sum_{j=1}^{3} \sum_{\substack{\alpha \\ \alpha \neq \gamma}}^{n_p} \left(\frac{\partial^2 l_{\alpha\gamma}}{\partial (q^{j\gamma})^2} \frac{\partial}{\partial l_{\alpha\gamma}} + \sum_{\substack{\beta \\ \beta \neq \gamma}}^{n_p} \frac{\partial l_{\alpha\gamma}}{\partial q^{j\gamma}} \frac{\partial l_{\beta\gamma}}{\partial q^{j\gamma}} \frac{\partial^2}{\partial l_{\alpha\gamma} \partial l_{\beta\gamma}} \right) \omega$$
$$= \sum_{\substack{\alpha \\ \alpha \neq \gamma}}^{n_p} \left(\frac{2}{l_{\alpha\gamma}} \frac{\partial}{\partial l_{\alpha\gamma}} + \sum_{\substack{\beta \\ \beta \neq \gamma}}^{n_p} \vartheta_{\alpha\gamma\beta\gamma} \frac{\partial^2}{\partial l_{\alpha\gamma} \partial l_{\beta\gamma}} \right) \omega = \sum_{\substack{\alpha \\ \alpha \neq \gamma}}^{n_p} \left(\frac{2}{l_{\alpha\gamma}} \frac{\partial}{\partial l_{\alpha\gamma}} + \frac{\partial^2}{(\partial l_{\alpha\gamma})^2} + \sum_{\substack{\beta \\ \beta \neq \gamma \\ \alpha \neq \beta}}^{n_p} \vartheta_{\alpha\gamma\beta\gamma} \frac{\partial^2}{\partial l_{\alpha\gamma} \partial l_{\beta\gamma}} \right) \omega$$

where

(6.94a) $$\vartheta_{\alpha\gamma\beta\gamma} \equiv \sum_{j=1}^{3} \frac{(q^{j\alpha} - q^{j\gamma})(q^{j\beta} - q^{j\gamma})}{l_{\alpha\gamma} l_{\beta\gamma}} \equiv \cos \varphi_{\alpha\gamma\beta\gamma}$$

(6.94b) $$\vartheta_{\alpha\gamma\alpha\gamma} = 1 \;\forall\; \alpha \neq \gamma; \quad \vartheta_{\alpha\gamma\beta\gamma} = \vartheta_{\beta\gamma\alpha\gamma} = \vartheta_{\gamma\alpha\gamma\beta} = \vartheta_{\gamma\beta\gamma\alpha}; \quad \vartheta_{\alpha\gamma\beta\gamma} \neq 0 \;\forall\; \alpha \neq \gamma \wedge \beta \neq \gamma$$

and $\varphi_{\alpha\gamma\beta\gamma}$ is the angle between the lines $\overrightarrow{\alpha\gamma}$ and $\overrightarrow{\beta\gamma}$.

For example if $n_p = 3$ and $\gamma = 1$ then (6.93) gives

(6.95a) $$\nabla_1^2 \omega = \left(\frac{2}{l_{21}} \frac{\partial}{\partial l_{21}} + \frac{2}{l_{31}} \frac{\partial}{\partial l_{31}} + \frac{\partial^2}{\partial (l_{21})^2} + \frac{\partial^2}{\partial (l_{31})^2} + 2 \cos \varphi_{3121} \frac{\partial^2}{\partial (l_{31}) \partial (l_{21})} \right) \omega$$

where

(6.95b) $2l_{21}l_{31}\cos\varphi_{3121} = l_{21}^2 + l_{31}^2 - l_{32}^2$

The appropriate version of (6.88) is

(6.96) $$\left(\frac{\nabla_1^2}{m_1} + \frac{\nabla_2^2}{m_2} + \frac{\nabla_3^2}{m_3}\right)^2 \omega = 0$$

which can be written out explicitly by permuting the pairs of suffices in (6.95a). Consider the simpler PDE

(6.97) $$\left(\frac{\nabla_1^2}{m_1} + \frac{\nabla_2^2}{m_2} + \frac{\nabla_3^2}{m_3}\right)\omega = 0$$

Solutions of (6.97) are also solutions of (6.96). Now suppose that ω is assumed to be a function of a single length $l \equiv l_{21} = l_{12}$ (say). Then (6.97) reduces to

(6.98) $$\left(\frac{1}{m_1}\left[\frac{2}{l}\frac{d}{dl} + \frac{d^2}{dl^2}\right] + \frac{1}{m_2}\left[\frac{2}{l}\frac{d}{dl} + \frac{d^2}{dl^2}\right]\right)\omega = 0 \Rightarrow \left(\frac{2}{l}\frac{d}{dl} + \frac{d^2}{dl^2}\right)\omega = 0$$

with solution

(6.99) $$\omega = \frac{c_{21}}{l_{21}} + c_{00}; \quad c_{21}, c_{00} \text{ are constants}$$

Because (6.97) is linear the sum of all such solutions is also a solution of (6.96/97)

(6.100) $$\omega = \frac{c_{21}}{l_{21}} + \frac{c_{31}}{l_{31}} + \frac{c_{32}}{l_{32}} + c_{00}; \quad c_{21}, c_{31}, c_{32}, c_{00} \text{ are constants}$$

It is obvious from (6.93/94) that a generalisation of (6.100)

(6.100a) $$\omega = \sum_{\substack{\alpha,\beta \\ \alpha>\beta}}^{n_p} \frac{c_{\alpha\beta}}{l_{\alpha\beta}}; \quad c_{\alpha\beta} = c_{\beta\alpha} \text{ are constants}$$

is always a solution of (6.88). There is nothing in the equations that prevents us from defining

(6.100b) $c_{\alpha\beta} \equiv \mathsf{G} m_\alpha m_\beta$; see Newton's law of gravity [3], [10] But, equally, there is nothing in the equations that requires (6.100b). Strictly, the m_α, as they appear in (6.100 b), are gravitational masses; whereas, as they appear in the Hamiltonian (6.86), they are inertial masses.

The solutions (6.100/100a) are not, of course, the most general solutions of the PDEs (6.96 /88); the general solutions will contain additional terms. But (6.100) does show that pairs of particles are associated with an inverse distance potential term as in (6.53). In general ω is the sum of a regular function of the $l_{\alpha\beta}$ and one or more singular terms. The regular function has the property that it becomes constant as *all* the $l_{\alpha\beta} \to 0$. The singular terms have the property that if *any* of the associated $l_{\alpha\beta} \to 0$ then $\omega \to \infty$; that is, a singularity is a symptom of the existence of two or more particles. Singular terms like those at (6.100) generate an inverse square force, on a pair of particles, in the direction of the line joining them. This dominates if the particles are close.

It might be, of course, that singularities of the kind (6.100) are not the only sort possible. The solutions of (6.93) might contain singularities generated by inverse powers or products of the $l_{\alpha\beta}$ of order greater than one. At short distances such singularities could dominate those of the kind (6.100); in that case the force between close particles would not be inverse square. This embarrassing eventuality seems unlikely; but it also seems tricky to prove that it is impossible!

6.12 Introduction Of A Test Particle When $n_p > 1$ And C Is Flat

When discussing gravity we often suppose that the field is sensed by a test particle of infinitesimal mass. In GR the field is characterised by the way adjacent geodesics of such test particles deviate from one another (equation of geodesic deviation [9], p. 90]). Suppose that we introduce such a test particle in addition to the system of particles of finite mass contained in P. Thus the number of particles increases from n_p to $n_p + 1$. Let the infinitesimal mass be m_γ where $\gamma = n_p + 1$. The PDE for ω then reads

(6.101) $\left(\frac{\nabla_\gamma^2}{m_\gamma} + \sum_\alpha^{n_p} \frac{\nabla_\alpha^2}{m_\alpha} \right)^2 \omega = 0; \quad m_\alpha >> m_\gamma \ \forall\, \alpha \in [1, n_p]; \quad \gamma = n_p + 1$; see (6.88)

Multiply by m_γ^2 and we are left with the approximate PDE

(6.102) $\nabla_\gamma^2 (\nabla_\gamma^2 \omega) \to 0$; see (6.25b/27)

where

(6.102a) $$\nabla_\gamma^2\omega = \sum_{\alpha}^{n_p}\left(\frac{2}{l_{\alpha\gamma}}\frac{\partial}{\partial l_{\alpha\gamma}} + \frac{\partial^2}{(\partial l_{\alpha\gamma})^2} + \sum_{\substack{\beta \\ \alpha\neq\beta}}^{n_p}\vartheta_{\alpha\gamma\beta\gamma}\frac{\partial^2}{\partial l_{\alpha\gamma}\partial l_{\beta\gamma}}\right)\omega; \quad \gamma = n_p + 1; \text{ see (6.93)}$$

which has solutions of the form $\omega(l_{\alpha\gamma})\ \forall\ \alpha\in[1,n_p]$. In fact

(6.103) $\omega = \omega_0(l_{\alpha\beta}) + \Delta\omega(l_{\alpha\gamma})\ \forall\ \alpha,\beta; \quad \gamma = n_p + 1 \in [1, n_p]$

where

(6.103a) $$\left(\sum_{\alpha}^{n_p}\frac{\nabla_\alpha^2}{m_\alpha}\right)^2\omega_0 = 0; \quad \nabla_\gamma^2(\nabla_\gamma^2\Delta\omega)\to 0; \quad \gamma = n_p + 1; \text{ see (6.88/93)}$$

Thus, despite the reciprocal dependence on mass exhibited at (6.101), the introduction of the test particle has no effect on the potential.

7. The Vector Potentials F^j And The Scalar Potential V

From their positions in the RHS of (3.18)/ (3.19a) we would describe the $f^j \to F^j$ and $v \to V$ as *gauge potentials*. In what follows we discuss their formal properties in the EM case.

7.1 Identification Of The Coefficients F^j In The EM Case

If (4.1) holds then the path of X in C is a geodesic. But if

(4.5) $f^j \neq 0\ \exists\ j \wedge/\vee\ v \neq 0$

then the path of X in C is not a geodesic. The classical tensor equation of motion (derived by eliminating the momenta from Hamilton's equations acting on (3.19a)) is

(7.1) $$\begin{aligned}&\ddot{q}^m + \Gamma^m_{kl}\dot{q}^k\dot{q}^l = \\ &-g^{um}_{,v}\dot{q}^v f_u - g^{um}\left(f^j_{,u}(\dot{q}_j - f_j) - \frac{1}{2}g^{jk}_{,u}(\dot{q}_j f_k + \dot{q}_k f_j - f_j f_k) + 2\mathrm{K}v_{,u}\right) + f^m_{,v}\dot{q}^v\end{aligned}$$

where, of course, C may be curved. Now consider a single particle in Minkowskian P and flat C. With Galilean coordinates the metric of $\mathrm{C}\equiv\mathrm{P}$ is

(7.2) $g_{jk} = 0,\ j \neq k,\ g_{jj} = g^{jj} = -1, -1, -1, +1$

and, because the g^{uv} are constant, (7.1) simplifies to

(7.3) $\ddot{q}^m = -g^{mm}\left(f^j_{,m}(\dot{q}_j - f_j) + 2\mathrm{K}v_{,m}\right) + f^m_{,v}\dot{q}^v$; not summed on m

Substituting the metric (7.2) into (7.3)

(7.4)
$$\begin{aligned}&\ddot{q}^\lambda = \dot{q}^j(f_{j,\lambda} - f_{\lambda,j}) - f^j_{,\lambda}f_j + 2\mathrm{K}v_{,\lambda}; \quad \dot{q}^\lambda = -\dot{q}_\lambda; \quad \dot{q}^4 = \dot{q}_4; \quad f^\lambda = -f_\lambda; \quad f^4 = f_4\\ &\ddot{q}^4 = -\dot{q}^j(f_{j,4} - f_{4,j}) + f^j_{,4}f_j - 2\mathrm{K}v_{,4}\end{aligned}$$

where the Greek indices run from 1 to 3 and the Roman indices run from 1 to 4.

We now compare these equations of motion with those of a single charged particle in an EM field

(7.5) $$c\frac{d(m\underline{\dot{q}})}{dq^4} = \frac{\varepsilon}{c}\left(\underline{u}\times\underline{b} + c\underline{e}\right), \quad c\frac{d(m\dot{q}^4)}{dq^4} = \varepsilon\underline{e}.\underline{u}; \quad \underline{b} \equiv \nabla\times\underline{a}; \quad \underline{e} \equiv -\nabla\phi - \frac{\partial\underline{a}}{\partial q^4}$$ [5]

where: m is the mass, ε is the charge, q^4/c is the proper time, $\underline{\dot{q}}$ is the proper 3-velocity, $\underline{u}$ is the 3-velocity, $\underline{a}$ is the 3-vector potential, ϕ is the scalar potential, $\underline{b}$ is the 3-magnetic induction, $\underline{e}$ is the 3-electric intensity. These SR equations have received extensive experimental verification. Written in coordinate form (7.5) is

(7.6)
$$\begin{aligned}&c\frac{d(m\dot{q}^\lambda)}{dq^4} = \frac{\varepsilon}{c}\left(u^\mu(a_{\mu,\lambda} - a_{\lambda,\mu}) - u^4(\phi_{,\lambda} + a_{\lambda,4})\right)\\ &c\frac{d(m\dot{q}^4)}{dq^4} = -\frac{\varepsilon}{c}\left(u^\mu(a_{\mu,4} + \phi_{,\mu})\right) \quad u^j \equiv c\frac{dq^j}{dq^4} \Rightarrow u^4 = c\end{aligned}$$

Multiply through by dq^4/cdt and we get

(7.7)
$$\begin{aligned}&\ddot{q}^\lambda = \frac{\varepsilon}{mc}\left(\dot{q}^\mu(a_{\mu,\lambda} - a_{\lambda,\mu}) - \dot{q}^4(\phi_{,\lambda} + a_{\lambda,4})\right)\\ &\ddot{q}^4 = -\frac{\varepsilon}{mc}\left(\dot{q}^\mu(a_{\mu,4} + \phi_{,\mu})\right) \quad u^j \equiv c\frac{dq^j}{dq^4} \Rightarrow u^4 = c\end{aligned}$$

These equations can be compared, directly, with (7.4) by regarding the $\underline{q}^j$ as arbitrary. The comparison yields

(7.8)
$$f_{\mu,\lambda} - f_{\lambda,\mu} = \frac{\varepsilon}{mc}(a_{\mu,\lambda} - a_{\lambda,\mu}); \quad f_{4,\lambda} - f_{\lambda,4} = -\frac{\varepsilon}{mc}(\phi_{,\lambda} + a_{\lambda,4});$$
$$f^j_{,\lambda} f_j = 2\mathrm{K} v_{,\lambda}; \quad f^j_{,4} f_j = 2\mathrm{K} v_{,4}$$

The first three of these equations relate the 4-curl of the f_j to curl $\underline{a}$ and grad ϕ. But the last two equations relate the gradient of v to $f^j_{,k} f_j$. If we define

(7.9) $$4\mathrm{K}v \equiv f^j f_j \Rightarrow 4\mathrm{K}v_{,k} \equiv f^j_{,k} f_j + f^j f_{j,k} = 2 f^j_{,k} f_j \Rightarrow f^j_{,k} f_j = 2\mathrm{K} v_{,k}$$

then we get a result consistent with (7.8). In consequence (3.19a) can be written

(7.10) $$H \equiv \mathrm{K}\left(p^u + \frac{f^u}{2\mathrm{K}}\right)\left(p_u + \frac{f_u}{2\mathrm{K}}\right)$$

Taking account of (7.8), with appropriate definitions, this is

(7.11) $$H = \frac{1}{2m}\left(-\sum_{\mu=1}^{3}\left(p_\mu + \frac{\varepsilon}{c}a_\mu\right)^2 + \left(p_4 - \frac{\varepsilon}{c}\phi\right)^2\right); \quad \mathrm{K} \equiv \frac{1}{2m}; \quad f_\mu \equiv \frac{\varepsilon}{mc}a_\mu; \quad f_4 \equiv -\frac{\varepsilon}{mc}\phi$$

which can be shown to give the equations of motion (7.7) directly. Notice that the definitions (of the f_j in terms of the a_μ and ϕ) are sufficient but they may not be necessary. This identification of the f_j is classical because it is based on (3.19a).

7.2 The Impact Of The Theta Equation On Maxwell's Equations

When there is but a single particle in P, and $\mathrm{C} \equiv \mathrm{P}$ is Minkowskian with Galilean coordinates, the metric is

(7.12) $$g^{jk} = 0,\ j \neq k,\ g^{jj} = g_{jj} = -1, -1, -1, +1; \quad n_p = 1; \quad n_d = 4$$

The theta equation then becomes

(7.13) $$\Box^2(\Box^2\theta) = 0; \quad \Box^2 \equiv -\nabla^2 + \frac{\partial^2}{\partial(q^4)^2}; \quad q^4/c \text{ is coordinate time}$$

Notice that solutions of

(7.14) $\Box^2\theta = 0$; standard wave equation [11]

are also solutions of (7.13). Because the f^j are candidates for θ these results require that

(7.15) $\Box^2(\Box^2\underline{a}) = \underline{0}$; $\Box^2(\Box^2\phi) = 0$; see (7.11)

We consider the impact of (7.15) on Maxwell's equations in more detail, as follows. With 3-vector notation Maxwell's equations in the *Lorentz gauge* can be written as follows [12]:

$$(7.16)\quad a)\ \nabla.\underline{e} = 4\pi\rho;\quad b)\ \nabla\times\underline{e} = -\frac{\partial \underline{b}}{\partial q_4};\quad c)\ \nabla.\underline{b} = 0;\quad d)\ \nabla\times\underline{b} = \frac{4\pi \underline{j}}{c} + \frac{\partial \underline{e}}{\partial q_4}$$

$$(7.17)\quad a)\ \nabla.\underline{a} + \frac{\partial\phi}{\partial q^4} = 0;\quad b)\ \nabla.\underline{j} + c\frac{\partial\rho}{\partial q^4} = 0;\quad c)\ \Box^2\underline{a} = \frac{4\pi}{c}\underline{j};\quad d)\ \Box^2\phi = 4\pi\rho$$

where

$$(7.18)\quad \underline{b} \equiv \nabla\times\underline{a};\quad \underline{e} \equiv -\nabla\phi - \frac{\partial \underline{a}}{\partial q^4}\ \text{given the Lorentz condition (7.17a)}$$

Given (7.16a/d), (7.17a) and (7.18) as definitions, five of the eight equations (7.16/17) can be deduced as identities. The first two of the vector identities [12]

$$(7.19)\quad \nabla\times(\nabla\phi) = \underline{0};\quad \nabla.(\nabla\times\underline{a}) = 0;\quad \nabla\times(\nabla\times\underline{a}) = \nabla(\nabla.\underline{a}) - \nabla^2\underline{a}$$

show that (7.16b/c) follow directly from (7.18). Regarding (7.16a) as a definition of ρ result (7.17 d) follows from (7.17a). Regarding (7.16d) as a definition of $\underline{j}$ result (7.17b) follows from (7.19) and (7.16a); similarly result (7.17c) follows from (7.16d), (7.17a) and the last of the identities (7.19). We can also show that the components of the force fields satisfy the wave equation [11]

$$(7.20)\quad \Box^2\underline{e} = \underline{0};\quad \Box^2\underline{b} = \underline{0}$$

We deduce from (7.15) and (7.17c/d) that

$$(7.21)\quad \Box^2\underline{j} = \underline{0};\quad \Box^2\rho = 0$$

That is the new equations (7.15) require that the current and charge densities satisfy the standard wave equation. This result seems, at first sight, to be contrary to experience. The equations (7.21) require that any disturbance in the current density $\underline{j}$ or the charge density ρ shall be propagated with speed c. But Dirac's theory of the electron/ positron says that the instantaneous speed (of an electron/ positron), *measured along any Cartesian axis*, is always $\pm c$ [1]. This is because Dirac's speed operators (the $c\alpha$ s) do not commute either with the Hamiltonian or with each other and have only the eigenvalues $\pm c$. So, keeping in mind that the equations (7.16/17) represent a cloud of charged particles as an idealized, continuous, classical fluid, the equations (7.21) may not be as contrary as might first appear. Note that the theta equation is a quantum equation; it may therefore appear in quantum mechanical arguments along side Dirac's equation.

If, of course, $\underline{j} = \underline{0}$ and $\rho = 0$ then the space is empty of charge, (7.17c/d) require that the a^j satisfy the wave equation and (7.15) is satisfied trivially; the corresponding particles must be photons. Keeping in mind that, the original rudimentary classical model, from which the constraints are derived, has structureless point particles moving in an otherwise empty space, it seems the assumptions $\underline{j} = \underline{0}$ and $\rho = 0$ are appropriate. The Maxwell equations with $\underline{j} \neq \underline{0}$ and $\rho \neq 0$ appear to be essentially classical.

8. Notes On The ANPA References

[19] This reference is a general paper on time. It does not consider the relativistic properties of time because to do so, before an ANPA audience, would be to 'carry coals to Newcastle'! It does, however, consider quantum time and the phenomenon of chaos.

[20] The work on Constraints Theory, together with its implications, was first begun in this reference. No formal proof, that constraint 1 requires the Hamiltonian to be a polynomial of order not exceeding 2, is given; but examples strongly suggest this. Various proofs have been tried. The proof given in Section 3 is favoured because it does not assume, ab initio, that H is a polynomial. The Theta Equation is not derived in [20] but, again, examples foreshadow it.

[21] Much of the work described above is sketched in graphical form along with some textual argument.

[22] This sprawling paper is a flawed effort to explain some of the structures of CM by assuming that the measurement of rate is an averaging process. Dirac [1] invoked this idea, successfully, to explain how his velocity operators (eigenvalues $\pm c$) give rise to observations subject to the classical SR relations governing velocity and momentum. These notions pertain to Constraints Theory because it can be shown that linear Hamiltonians, with constant *matrix* coefficients (of finite order), satisfy the constraints to, at least, level 3. The matrices, of course, have discrete eigenvalues; and all but one of them are velocity operators. Considerable effort has been focussed, subsequent to [22], on related topics. But the results are not definite; and so they are not reported here. Some of the work on linear Hamiltonians is given in drafts 1 and 2 of An Overview of Constraints Theory (progenitors of this document). The references [16,17,18] appear in these documents.

9. Linear Hamiltonians With Matrix Coefficients- A New Chapter?

The only definite result concerning linear Hamiltonians, with constant matrix coefficients, is that they satisfy the constraints, without caveats, at least to level 3. They are of importance because the Dirac equation derives from a linear Hamiltonian with matrix coefficients that are *either functions of the coordinates or constant*. One complication that arises, in the investigation of such Hamiltonians in Constraint Theory, is that, ostensibly, every matrix coefficient is a candidate for θ. To study them is to take us outside our self-imposed remit of deriving CM from QM. Nevertheless, there are partial results. A topic that needs investigation in relation to Hamiltonians, in general, and linear Hamiltonians, particular, is that of *constants of the motion*. For example, the angular momentum operators (see Section 2.10 above) are constants of the Dirac Hamiltonian only if spin operators are added. The significance of constants of the motion in QM is that, because they commute with the Hamiltonian, they can be averaged, in the course of measurement, for an indefinite time. They therefore exhibit classical permanence and structure.

Perhaps, if we can get together enough definite results, we should add a chapter on linear Hamiltonians with matrix coefficients. My feeling is that, although slippery, they may provide, through Constraint Theory, a strong explanation of the physical importance of Euclidean 3-space and Minkowskian 4-space; see Section 6.2 above.

A. M. Deakin 18/11/2008

10. References

[1]	Dirac, P. A. M.	‘The Principles Of Quantum Mechanics’, 4^{th} Ed., OUP, 1958
[2]	Jordan, T. F.	‘Linear Operators For Quantum Mechanics’, John Wiley, 1969
[3]	Misner, C. W. Thorne, K. S. Wheeler, J. A.	‘Gravitation’, W.H. Freeman, 1973
[4]	Temple, G.	‘The General Principles of Quantum Theory’, Methuen, 1956
[5]	Leech, J. W.	'Classical Mechanics', Methuen, 1958
[6]	Lovelock, D. Rund, H.	‘Tensors, Differential Forms And Variational Principles’, Dover, 1989
[7]	Eddington, A. S.	‘Mathematical Theory Of Relativity’, Chelsea, 1975
[8]	Spain, B.	‘Tensor Calculus’, Dover, 2003
[9]	Synge, J. L. Schild, A.	‘Tensor Calculus’, Dover, 1969
[10]	Grossman, N.	‘The Sheer Joy Of Celestial Mechanics’, Birkhäuser, 1996
[11]	Coulson, C. A.	‘Waves’, Oliver & Boyd, 1958
[12]	Rutherford, D. E.	‘Vector Methods’, Oliver & Boyd, 1957
[13]	Kauffman, L. H.	‘Non-commutative worlds’, New Journal of Physics, 2004
[14]	Kilmister, C.W.	‘Hamiltonian Dynamics’, Longmans, 1964
[15]	Kilmister, C.W.	‘Lagrangian Dynamics’, Logos, 1967
[16]	Landau, L. D. Lifshitz, E. M.	‘Quantum Mechanics’, p.132, Pergamon, Oxford, 1965
[17]	Arfken, G.	‘Mathematical Methods For Physicists’, 2nd Ed., Academic Press, 1970
[18]	Aitken, A. C.	‘Determinants And Matrices’, Oliver & Boyd, 1958
[19]	Deakin, A. M.	‘Time’, p. 44, Proc. ANPA 25, 2004

[20]	Deakin, A. M.	'Where Does Schrodinger's Equation Really Come From?', p. 277, Proc. ANPA 20, 1999
[21]	Deakin, A. M.	'Classical Mechanics From Quantum Mechanics Via Commutators', p. 115. Proc. ANPA 28, 2006
[22]	Deakin, A. M.	'Can We Run Dirac's Argument Backwards', p. 59, Proc. ANPA 23, 2001
[23]	Cottingham, W. N. Greenwood, D. A.	'An Introduction to the Standard Model of Particle Physics', 2nd Ed., CUP, 2007
[24]	Feynman, R. P. Hibbs, A. R.	'Quantum Mechanics and Path Integrals', McGraw-Hill, 1965
[25]	Deakin, A. M.	'Kilmister Equ. 1- Notes', MSWord Document
[26]	Deakin, A. M.	'Kilmister Equ. 1- Calculations', Maple Document